Time evolution of a modified Feynman ratchet with velocity-dependent fluctuations and the second law of thermodynamics

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Abstract: It is shown that the randomness of Brownian motion at thermodynamic equilibrium can be spontaneously broken by velocity-dependence of fluctuations, i.e., by dependence of values or probability distributions of fluctuating properties on Brownian-motional velocity. Such randomness-breaking can spontaneously obtain via interaction between Brownian-motional Doppler effects — which manifest the required velocity-dependence — and system geometrical asymmetry. A *non*random walk is thereby spontaneously superposed on Brownian motion, resulting in a systematic net drift velocity *despite* thermodynamic equilibrium. (By contrast, the mere existence of fluctuations, without velocity-dependence thereof, can*not*, in general, effect such randomness-breaking.) The time evolution of this systematic net drift velocity — implying acceleration — is derived for the velocity-dependent modification of Feynman's ratchet developed in this paper. On this basis, we derive the time evolution of the velocity probability density, of the accelerating force, and of power output = (accelerating force) × (systematic net drift velocity). Quantitative results are obtained for all times, including final steady-state values. We show that said (a) spontaneous randomness-breaking, and (b) consequent systematic net drift velocity, imply: (c) bias from the Maxwellian of the system's velocity probability density, (d) the force that tends to accelerate it, and (e) its power output. Maximization of (a) through (e) above, especially of (e) power output, is discussed. Uncompensated decreases in total entropy, challenging the second law of thermodynamics, are implied by (a) through (e) above.

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1. Introduction: The Zhang formulation of the second law of thermodynamics, and a velocity -dependent modified Feynman-ratchet model

The Zhang [1] formulation of the second law of thermodynamics (second law) states that no spontaneous momentum flow is possible in an isolated system. By *spontaneous*, it is meant [1]: not merely (a) *sustaining*, i.e., permanent; but also (b) *robust*, i.e., capable of withstanding dissipation, of surviving disturbances, and of generating (regenerating) itself if initially nonexistent (if destroyed). The Zhang [1] formulation of the second law implies that, *at thermodynamic equilibrium (TEQ)*, not even *merely sustaining* momentum flow is possible, i.e., that *no* systematic motion — most generally, *no* systematic process — is possible *at TEQ*: Systematic processes generated and maintained *spontaneously despite TEQ* violate the second law; by contrast, systematic *merely sustaining*, i.e., *non*robust *and non*dissipative — and hence *non*spontaneous — processes do not violate the second law, but merely imply that TEQ has *not* been *completely* realized [1,2]. [Given any irreversibility (e.g., friction), (*non*spontaneous) *merely sustaining* processes lose *even their sustainability* — they become *non*robust *and dissipative* — their negentropy (and hence free energy) is lost, and TEQ *is completely* realized [1,2].]

Corollary: The Zhang [1] formulation of the second law implies that, at *non*-TEQ, spontaneous momentum flow is *possible* in an isolated system; and hence that, at *non*-TEQ, systematic motion — most generally, systematic process — is *possible*. Said *possibility* may be *actualized* or *in potentiality*: Example: a gas constrained to within less than the total volume of its container is at *non*-TEQ and has the *potential* for systematic motion — expansion — which is *actualized* upon release of the constraint.

Feynman's classic ratchet and pawl [3] elucidates the Zhang [1] formulation of the second law: In the original classic "Ratchet and Pawl" chapter [4], it is stated, as the upshot concerning Feynman's classic system,

> "In spite of all our cleverness of lopsided design, if the two temperatures are exactly equal there is no more propensity to turn one way than the other. The moment we look at it, it may be turning one way or the other, but in the long run it gets nowhere. The fact that it gets nowhere is really the fundamental deep principle on which all of thermodynamics is based."

Recently, various formulations of the second law have been challenged, in both the quantum [5–8] and classical [5,8–10] regimes.

In this paper [11], we show that velocity-dependent fluctuations (but not fluctuations in general) challenge the second law in the classical regime. [A digression concerning limited aspects of the quantum regime is provided in Sect. 6. Otherwise, except for the last four paragraphs of Sect. 3 (and a few very brief mentions elsewhere), this paper deals only with the classical regime.] Our challenge is most self-evident with respect to the Zhang [1] formulation of the second law, but (as will be discussed in the last four paragraphs of Sect. 3): The Zhang [1] formulation of the second law is *maximally* strong — no other formulation thereof can be stronger [although some other(s) may be equally strong]. [Classically (with one exception [6gg] that is not applicable insofar as this present paper is concerned) — all formulations of the second law are equivalent — but not so quantum-mechanically [6s–6ff].] Hence: A challenge to the Zhang [1] formulation of the second law is also a challenge to *all* other formulations thereof.

In this paper [1], Feynman's ratchet [3] is modified to the minimum extent necessary to ensure that velocity-dependence of fluctuations can spontaneously break the randomness of its Brownian motion at

TEQ — spontaneously superposing a *non*random walk on its Brownian motion and hence challenging the second law. This minimally-modified Feynman ratchet, illustrated in Fig. 1, will now be described.

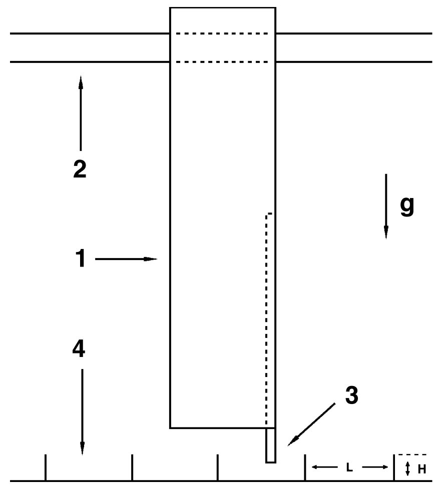

Figure 1. Modified Feynman ratchet with velocity-dependent fluctuations.

In the right-handed Cartesian coordinate system of Fig. 1, the $+X$, $+Y$, and $+Z$ directions are to the right, into the page, and upwards, respectively. The Brownian motion of the disk 1 of mass m' (shown edge-on in Fig. 1) is constrained to be X-directional by the frictionless guide 2. The pawl 3 of mass m (whose lower tip protrudes below the disk in Fig. 1) is in a vertical groove within the $+X$ disk face, wherein — in addition to its X-directional Brownian motion in lockstep with the disk as part of the combined disk-and-pawl system (DP) — it also has Z-directional Brownian motion relative to the disk per se. The DP's total mass is $M = m' + m \gg m$. Each peg 4 is of Z-directional height H, and is separated from adjacent pegs by X-directional distance L. The pawl's altitude Z is the vertical distance of its undersurface above the $Z = 0$ level at the floor of the peg row 4, and is restricted to $Z \geq Z_{\rm min}$ ($0 < Z_{\rm min} < H$) by a stop within the $+X$ disk face. (A simple design for the stop: Let the vertical groove that accommodates the pawl have thinner slots extending in the $+Y$ and $-Y$ directions. These slots accommodate pins extending from the pawl in the $+Y$ and $-Y$ directions, respectively. The floors of these slots preclude Z-directional motion of the pins below the pin/slot-floor contact level, thereby restricting the pawl's altitude to $Z \geq Z_{\rm min}$.) The *net* peg height is thus $H_{\rm net} \equiv H - Z_{\rm min}$ ($0 < H_{\rm net} < H$). The entire system, including the DP, is at TEQ with equilibrium blackbody radiation (EBR) at temperature T. L is, for simplicity, taken to be large compared with the combined pawl-plus-peg X-directional thickness (see Appendix A); yet L can easily still be small enough so that changes in the DP's X-directional Brownian-motional velocity V occur, essentially, *only at* pawl-peg bounces, and *not* via DP-EBR X-directional momentum exchanges *between* pawl-peg bounces (see Appendix B) [12]. (The frictionless guide 2, of course, has no effect on V.) A uniform gravitational field g is attractive downwards (in the $-Z$ direction). The $V = 0$ rest frame — wherein (a) the frictionless guide 2 and peg row 4 are fixed and (b) the EBR at temperature T is isotropic — is (for simplicity) taken as that of g's source [of mass $\gg M$ (or even $\ggg M$)]. Except for the EBR, our system is nonrelativistic: i.e., all speeds (except of EBR photons) are $\ll c$, and all pertinent differences in gravitational potential (e.g., gH) are $\ll c^2$.

The right-handed Cartesian coordinate system described in the immediately preceding paragraph is the most appropriate one given *linear* X-directional DP Brownian motion. For transformation to *circular* X-

directional DP Brownian motion, said right-handed Cartesian coordinate system can be transformed into a right-handed cylindrical coordinate system by (a) curving the X-directional axis into a circle, and (b) letting the $+X$, $+Y$, and $+Z$ directions be counterclockwise, radially outwards from the center of this circle, and upwards, respectively.

Corresponding to X-directional Brownian-motional velocity V of the DP, to first order in V/c, Doppler-shifted EBR at temperature [13]

$$T_{\pm}(V,\alpha) = T\left(1 \pm \frac{V\cos\alpha}{c}\right) \tag{1}$$

impinges on the $\pm X$ disk face at angle α from the $\pm X$ direction — at a rate proportional both to the differential solid angle $2\pi\sin\alpha d\alpha$ and, by Lambert's cosine law, to $\cos\alpha$ [13]. {The pawl, being in the $+X$ disk face, "sees" EBR impinging — as per the immediately preceding sentence [including (1)] with the $+$ signs — *only* from directions with $+X$ components (except for its lower tip — of *negligible* size compared with the *entire* pawl even at maximum tip protrusion, i.e., even at $Z = Z_{\min}$ — when said tip protrudes below the disk).} Averaging over the range $0 \leq \alpha \leq \pi/2$ [13],

$$\begin{aligned} T_{\pm}(V) &= \langle T_{\pm}(V,\alpha)\rangle \\ &= \frac{\int_0^{\pi/2} T\left(1 \pm \frac{V\cos\alpha}{c}\right)\sin\alpha\cos\alpha d\alpha}{\int_0^{\pi/2}\sin\alpha\cos\alpha d\alpha} \\ &= T\left(1 \pm \frac{2V}{3c}\right). \end{aligned} \tag{2}$$

The DP's thermal response time is sufficiently short that $T_{+}(V)$ [$T_{-}(V)$] is the temperature, corresponding to V having a given value, of the $+X$ disk face (including the pawl) *itself* [12] [of the $-X$ disk face *itself* [12]] — *not* merely of Doppler-shifted EBR "seen" thereby [13].

The stop *within the $+X$ disk face* — and hence *itself* [12] at temperature, corresponding to V having a given value, of $T_{+}(V)$ [12,13] — restricts the pawl's altitude to $Z \geq Z_{\min}$: this prevents mechanical thermal contact [although not radiative thermal contact (which is *negligible*)] between the floor of the peg row — at elevation $Z = 0$ and temperature T — and the pawl's undersurface. (Except when the pawl's undersurface protrudes below the disk, the $+X$ disk face shields it from EBR impinging from directions with $-X$ components — and, in any case, the pawl's undersurface area is *negligible* compared with that of the *entire* pawl.) The pawl's thermal isolation within the $+X$ disk face is thereby improved — helping to ensure that $T_{+}(V)$ is the temperature, corresponding to V having a given value, of the pawl *itself* [12], *not* merely of Doppler-shifted EBR "seen" thereby [13].

In accordance with the Boltzmann distribution, and applying (2) with the $+$ signs, the *conditional* probability [14] $P(Z > H|V)$ that the pawl, of weight mg, can attain sufficient altitude $Z > H$ to jump the pegs

— and hence *not* to impede the DP's X-directional Brownian motion — given V, is

$$\begin{aligned} P(Z > H|V) &= \exp[-mg\,(H - Z_{\min})\,/kT_{+}(V)] \\ &\equiv \exp[-mgH_{\text{net}}/kT_{+}(V)] \\ &= \exp\left\{-mgH_{\text{net}} \Big/ \left[kT\left(1 + \frac{2V}{3c}\right)\right]\right\} \\ &\equiv \exp\left[-A \Big/ \left(1 + \frac{2V}{3c}\right)\right] \\ &= \left(1 + \frac{2AV}{3c}\right) e^{-A}. \end{aligned} \tag{3}$$

The second step of (3) restates the definition (initially given near the middle of the paragraph immediately following Fig. 1)

$$H_{\text{net}} \equiv H - Z_{\min}, \tag{4a}$$

the third step of (3) is justified by (2) with the $+$ signs, the fourth step of (3) defines

$$A \equiv mgH_{\text{net}}/kT, \tag{4b}$$

and the last step of (3), which is correct to first order in V/c, is justified because V is nonrelativistic, with $|V| \ll c$ for all values of $|V|$ that have nonnegligible probabilities of being equaled or exceeded.

By (3), $P(Z > H|V)$ is slightly greater when $V > 0$ than when $V < 0$. Hence, *despite TEQ*, the *velocity-dependence* of $P(Z > H|V)$ spontaneously superposes a *non*random walk in the $+X$ (Forward) direction on the DP's Brownian motion — challenging the second law.

Note that $T_{\pm}(V, \alpha)$, $T_{\pm}(V)$, Z, and $P(Z > H|V)$ manifest velocity-dependent fluctuations. By contrast, T, H, $Z_{\min}$, $H_{\text{net}} \equiv H - Z_{\min}$, L, m', m, $M = m' + m \gg m$, g, and $A \equiv mgH_{\text{net}}/kT$ are parameters, fixed in any one given (thought) experiment.

2. Markovian time evolution and challenges to the second law

The derivation of our system's time evolution will be easiest if we first consider, in (5) – (22) and the associated discussions [except in defining notation in the third paragraph of this Sect. 2, and in the last step of (7)], *only* occasions when $|V|$ happens to have any *one* given value, i.e., when $V = \pm|V|$. Subsequently, we will average over *all* $\pm|V|$ pairs, i.e., over *all* $|V|$.

By (3) and (4), we have, to first order in $|V|/c$, for the *conditional* probabilities [14] F and R of $Z > H$ obtaining given DP Brownian motion in, respectively, the Forward or $+X$ direction at $V = +|V|$ and Reverse or $-X$ direction at $V = -|V|$,

$$F \equiv P(Z > H|V = +|V|) \equiv P(>|+) = \left(1 + \frac{2A|V|}{3c}\right) e^{-A} \tag{5}$$

and

$$R \equiv P(Z > H|V = -|V|) \equiv P(>|-) = \left(1 - \frac{2A|V|}{3c}\right) e^{-A}, \tag{6}$$

respectively. The states $Z > H$, $Z < H$, $V = +|V| > 0$, and $V = -|V| < 0$ are denoted as $>$, $<$, $+$, and $-$, respectively. [Since Z and V are *continuous* random variables, the *point* values $Z = H$ and $V = |V| = 0$ each has *zero* probability measure of occurrence — and hence does not finitely contribute to any quantity integrated or averaged over any *finite* range of Z and V, respectively (e.g., over *all* Z and over *all* V, respectively).] Given $V = \pm|V|$, immediately preceding any pawl-peg interaction, the DP is in *one* of the four states $> +$, $> -$, $< +$, or $< -$; the former two states implying that this interaction will be a pawl-over-peg jump, and the latter two that it will be a pawl-peg bounce. Immediately following a jump (bounce), $\operatorname{sgn} V$ is unchanged (reversed).

We now study our system's time evolution, given $V = \pm|V|$, in discrete time-steps of $\Delta t = L/|V|$ that separate consecutive pawl-peg interactions, with time N immediately preceding the $(N+1)\,\mathrm{st}$ pawl-peg interaction. If a quantity Q or an average thereof is time-dependent, then its value at time N is indicated via a subscript N. Let $\langle Q\rangle_N$ ($\langle\langle Q\rangle\rangle_N$) denote the expectation value at time N of a quantity Q over any *one* given $\pm|V|$ pair ($\langle Q\rangle_N$ *itself* subsequently averaged over *all* $|V|$). {Notes: (a) All averages in this paper are, in this wise, either over any *one* given $\pm|V|$ pair or over *all* $|V|$, except: (i) the average $\langle T_\pm(V,\alpha)\rangle$ over α in (2) (denoted via enclosure within single angular brackets), and (ii) some of the averages in Sect. 6, and in the Appendixes and Footnotes. (b) Consistently with the fifth-to-the-last sentence (especially the last clause thereof) of the paragraph immediately following Fig. 1: The combined pawl-plus-peg X-directional thickness is $\ll L$; hence, the X-directional spatial, and temporal, intervals separating consecutive pawl-over-peg jumps are only negligibly greater [by said thickness, and (said thickness)$/|V|$, respectively] than those separating consecutive pawl-peg bounces (jump preceded or followed by bounce being the intermediate case). See Appendix A.}

TEQ, i.e., *maximum initial total entropy*, implies that *initially*, at $N = 0$,

$$\begin{aligned} &P(+)_0 = P(-)_0 = \frac{1}{2} \\ &\iff \langle V\rangle_0 = |V|\,[P(+)_0 - P(-)_0] = 0 \implies \langle\langle V\rangle\rangle_0 = 0. \end{aligned} \tag{7}$$

The expression in (7) for $\langle V\rangle_0$ is true for *all* $\pm|V|$ pairs, hence implying that for $\langle\langle V\rangle\rangle_0$. For all $N \geq 0$,

$$\begin{aligned} &\langle V\rangle_N = |V|\,[P(+)_N - P(-)_N] \\ &\iff P(\pm)_N = \frac{1}{2}\left(1 \pm \frac{\langle V\rangle_N}{|V|}\right). \end{aligned} \tag{8}$$

The second line of (8) is justified by $P(+)_N + P(-)_N = 1$ and by (7).

Given $V = \pm|V|$ and $P(+)_N + P(-)_N = P(>|+) + P(<|+) = P(>|-) + P(<|-) = 1$, said time evolution is a two-state discrete-time Markov chain [15] with (a) states $+$ and $-$; and (b) the following *conditional* transition probabilities:

$$P[(+)_N|(+)_{N-1}] = P(>|+) = F, \tag{9a}$$

$$P[(-)_N|(-)_{N-1}] = P(>|-) = R, \tag{9b}$$

$$P[(-)_N|(+)_{N-1}] = P(<|+) = 1 - F, \tag{9c}$$

and

$$P[(+)_N|(-)_{N-1}] = P(<|-) = 1 - R. \tag{9d}$$

Note that (9) – (18) are correct not only for the *specific* F and R given by the rightmost terms of (5) and (6), respectively, but also for *general* F and R that are *at most* functions of $|V|$ *only* — and hence *constant* for any *one* given $|V|$. [Of course, (1), (2), (7), and (8) are correct *in*dependently of *any* mention of F and R.]

Applying (9a), (9d), and $P(+)_N + P(-)_N = 1$, we obtain, for all $N \geq 0$, [15]

$$\begin{aligned}
P(+)_N &= FP(+)_{N-1} + (1-R)P(-)_{N-1} \\
&= FP(+)_{N-1} + (1-R)[1 - P(+)_{N-1}] \\
&= (F+R-1)P(+)_{N-1} + 1 - R \\
&= (F+R-1)[(F+R-1)P(+)_{N-2} + 1 - R] + 1 - R \\
&= (F+R-1)\{(F+R-1)[(F+R-1)P(+)_{N-3} + 1 - R] \\
&\quad + 1 - R\} + 1 - R \\
&= (F+R-1)^N P(+)_0 + (1-R)\sum_{j=0}^{N-1}(F+R-1)^j \\
&= (F+R-1)^N \left(\frac{1}{2}\right) + (1-R)\frac{1-(F+R-1)^N}{2-F-R} \\
&= \frac{2(1-R) - (F-R)(F+R-1)^N}{2(2-F-R)} \\
\Longrightarrow\ P(-)_N &= 1 - P(+)_N \\
&= \frac{2(1-F) + (F-R)(F+R-1)^N}{2(2-F-R)} \\
\Longrightarrow\ P(\pm)_N &= \frac{1}{2}\left\{1 \pm \frac{(F-R)[1-(F+R-1)^N]}{2-F-R}\right\}.
\end{aligned} \tag{10}$$

The second step and third-to-the-last step of (10) are justified by $P(+)_N + P(-)_N = 1$. In the third through sixth lines of (10), a recursion relationship is developed via repeated substitution. In the seventh step of (10), we applied the first line of (7) and standard summation of the geometric series in the sixth line of (10). [If $N = 0$, then: (i) This geometric series contains no terms and hence vanishes. (ii) $(F+R-1)^0 = 1$ is true throughout the range $-1 \leq F+R-1 \leq 1$ of $F+R-1$, with possible difficulty only at the *point* value $F+R-1 = 0$. But, since $(F+R-1)^0 = 1$ remains true even as $F+R-1 \longrightarrow 0^{\pm}$ *infinitesimally closely* (from both above and below) — by *continuity* we take $(F+R-1)^0 = 1$ even at the *point* value $F+R-1 = 0$. Note that, among indeterminate forms, perhaps x^0 alone is so well-behaved, maintaining a fixed well-defined unique finite value (1) even as $x \longrightarrow 0^{\pm}$ infinitesimally closely (from both above and below) — by contrast, for example, $\frac{x}{0} \longrightarrow \pm\infty$ as $x \longrightarrow 0^{\pm}$. Hence, $\lim_{x,y\longrightarrow 0} x^y = \lim_{x,y\longrightarrow 0}\left(e^{\ln x}\right)^y = \lim_{x,y\longrightarrow 0} e^{y\ln x} \overset{|y\ln x|\ll 1}{\longrightarrow} 1 + y\ln x \overset{|y\ln x|\ll 1}{\longrightarrow} 1$: if $x = F+R-1$ and $y = 0$, then the last two steps immediately preceding yield *exactly* 1 — *not* merely a limiting value of 1. For perhaps the most general approach pertinent to (10) of $F+R-1$ to 0 that is consistent with $(F+R-1)^0 = 1$ even at the *point* value $F+R-1=0$, let $x=a\,(F+R-1)$ and $y=b\,(F+R-1)^n=b\left(\frac{x}{a}\right)^n=\frac{b}{a^n}x^n$, where a, b, and n are arbitrary positive constants. Then $\lim_{x,y\longrightarrow 0} x^y=\lim_{x\longrightarrow 0} x^{\frac{b}{a^n}x^n}=\lim_{x\longrightarrow 0}\left(e^{\ln x}\right)^{\frac{b}{a^n}x^n}=\lim_{x\longrightarrow 0} e^{\frac{b}{a^n}x^n\ln x}=\lim_{x\longrightarrow 0}\left(1+\frac{b}{a^n}x^n\ln x\right)=1+\frac{b}{a^n}\lim_{x\longrightarrow 0}x^n\ln x=1+0=1$ (the last four steps immediately preceding being justified because $\lim_{x\longrightarrow 0}x^n\ln x=0$ by L'Hospital's Rule).]

Applying the first line of (8) and the last line of (10) yields, for all $N\geq 0$,

$$\begin{aligned}\langle V\rangle_N&=|V|\,[P(+)_N-P(-)_N]\\&=|V|\,(F-R)[1-(F+R-1)^N]/(2-F-R).\end{aligned}\tag{11}$$

By (11), $\langle V\rangle_N$ is antisymmetric in F and R; hence, with*out* loss of generality, we *always* take $F\geq R\implies\langle V\rangle_N\geq 0$ — e.g., as obtains for the *specific* F and R given by the rightmost terms of (5) and (6), respectively. The equality $F=R\implies\langle V\rangle_N=0$ obtains *only* given: (a) the *point* value $V=|V|=0$, which has *zero* probability measure of occurrence; and/or (b) $N=0$. Our challenge to the second law requires the *strict* inequality $F>R\implies\langle V\rangle_N>0$ *despite TEQ*, which obtains given $|V|>0$ and $N\geq 1$.

Direct calculation of $P(+)_N$ and $P(-)_N$ via (10) can be cumbersome. However, applying the second line of (11) — and then the antisymmetry of $\langle V\rangle_N$ as per the paragraph immediately following (11) — to the last line of (10) further simplifies the already simpler expression given by the second line of (8) [restated in the first line of (12)]:

$$\begin{aligned}&P(\pm)_N=P(V=\pm\,|V|)_N=\frac{1}{2}\left(1\pm\frac{\langle V\rangle_N}{|V|}\right)\\&\implies P(V)_N=\frac{1}{2}\left(1+\frac{\langle V\rangle_N}{V}\right)=P(V)_0\left(1+\frac{\langle V\rangle_N}{V}\right).\end{aligned}\tag{12}$$

The further simplification as per the second line of (12) [wherein $P(V)_0=\frac{1}{2}$ and $V=\pm\,|V|$ insofar as (5) – (22) and the associated discussions are concerned] is justified by said antisymmetry.

By (11), the final steady-state value of $\langle V\rangle_N$, i.e.,

$$\langle V\rangle_\infty=|V|\,(F-R)/(2-F-R),\tag{13}$$

is reached at $N=1$ if $F+R-1=0\iff 2-F-R=1$; i.e.,

$$\begin{aligned}\langle V\rangle_1&=|V|\,(F-R)\text{ for }\mathit{all}\ 0\leq F,R\leq 1\iff -1\leq F+R-1\leq 1\\&=\langle V\rangle_\infty\ \text{ if } F+R-1=0\iff 2-F-R=1.\end{aligned}\tag{14}$$

Hence, $P(V)_N$ of (12) manifests similar behavior. The completion of time evolution at $N=1$ if $F+R-1=0\iff 2-F-R=1$ obtains for *all* quantities studied in this paper. [In Sect. 4, we will show that, while allowing time evolution to $N\longrightarrow\infty$ *does* maximize $\langle V\rangle_N$ and $\left|P(V)_N-\frac{1}{2}\right|$, it does *not* correspond to maximizing the force that tends to accelerate the DP in the $+X$ direction, or to our *primary* objective of maximizing its power output and hence its time rate of negentropy production. Supplementary discussions will be given in Appendixes B and C.]

Now, define

$$\langle V\rangle_{N+\frac{1}{2}} \equiv \frac{1}{2}(\langle V\rangle_N + \langle V\rangle_{N+1})$$
$$= |V|\,(F-R)[2-(F+R)(F+R-1)^N]/[2(2-F-R)] \tag{15}$$

and

$$\langle \Delta V\rangle_{N+\frac{1}{2}} \equiv \langle V\rangle_{N+1} - \langle V\rangle_N$$
$$= |V|\,(F-R)(F+R-1)^N. \tag{16}$$

Let f be the force that tends to accelerate the DP in the $+X$ direction. By Newton's second law and (16), at the $N \longrightarrow N+1$ transition, i.e., at the $(N+1)\,\mathrm{st}$ pawl-peg interaction, we have

$$\langle f\rangle_{N+\frac{1}{2}} = M\,\langle \Delta V\rangle_{N+\frac{1}{2}}\Big/\,\Delta t$$
$$= M\,\langle \Delta V\rangle_{N+\frac{1}{2}}\Big/\,(L/\,|V|)$$
$$= (MV^2/L)(F-R)(F+R-1)^N. \tag{17}$$

The second step of (17) is justified because consecutive pawl-peg interactions are separated in time by $\Delta t = L/\,|V|$. Let P^* be the DP's power output (not to be confused with probability P). Applying (15) and (17), at the $N \longrightarrow N+1$ transition, i.e., at the $(N+1)\,\mathrm{st}$ pawl-peg interaction, we have

$$\langle P^*\rangle_{N+\frac{1}{2}} = \langle fV\rangle_{N+\frac{1}{2}}$$
$$= \langle f\rangle_{N+\frac{1}{2}}\,\langle V\rangle_{N+\frac{1}{2}}$$
$$= \frac{M\,|V|^3}{2L}\frac{(F-R)^2(F+R-1)^N[2-(F+R)(F+R-1)^N]}{2-F-R}. \tag{18}$$

The second step of (18) is justified because $\langle V\rangle_{N+\frac{1}{2}}$ of (15) is *in*dependent of which of the four DP states ($>+$, $>-$, $<+$, or $<-$) — and hence of the corresponding $(N+1)\,\mathrm{st}$ pawl-peg interaction (jump or bounce) — that happens to occur at the $N \longrightarrow N+1$ transition [16]. This justification is more fully developed in Appendix D.

For the *specific* F and R given by the rightmost terms of (5) and (6), respectively; (11), the second line of (12), (17), and (18), respectively become

$$\langle V\rangle_N = (2V^2/3c)A[1-(2e^{-A}-1)^N]/(e^A-1), \tag{19}$$

$$P(V)_N = \frac{1}{2}\left\{1+\left(\frac{2V}{3c}\right)\frac{A[1-(2e^{-A}-1)^N]}{e^A-1}\right\}$$
$$= \frac{1}{2}+\left(\frac{V}{3c}\right)\frac{A[1-(2e^{-A}-1)^N]}{e^A-1}, \tag{20}$$

$$\langle f \rangle_{N+\frac{1}{2}} = (4M\,|V|^3\,/3Lc)Ae^{-A}(2e^{-A}-1)^N, \tag{21}$$

and

$$\langle P^* \rangle_{N+\frac{1}{2}} = \left(\frac{8M\,|V|^5}{9Lc^2}\right)\frac{A^2e^{-A}(2e^{-A}-1)^N[1-e^{-A}(2e^{-A}-1)^N]}{e^A-1}. \tag{22}$$

Now, consider *all* $\pm|V|$ pairs, i.e., *all* V, and hence also *all* $|V|$; especially, consider fluctuations of V among *all* possible values. TEQ, i.e., *maximum initial total entropy*, implies that *initially*, at $N=0$,

$$\begin{aligned} &P(V)_0 = P(V)_{\mathrm{mw}} = (M/2\pi kT)^{1/2}e^{-MV^2/2kT} \\ &\Longrightarrow\ P(|V|)_0 = P(|V|)_{\mathrm{mw}} = 2P(V)_0 = 2P(V)_{\mathrm{mw}} = (2M/\pi kT)^{1/2}e^{-MV^2/2kT}, \end{aligned} \tag{23}$$

$P(V)_{\mathrm{mw}}$ [$P(|V|)_{\mathrm{mw}}$] being the one-dimensional Maxwellian probability density of V [$|V|$]. Letting $P(V)_0 = \frac{1}{2} \longrightarrow P(V)_0 = P(V)_{\mathrm{mw}}$ and $V=\pm|V| \longrightarrow$ continuous V in the second line of (12), and then applying (19), we obtain

$$\begin{aligned} P(V)_N &= P(V)_0\left(1+\frac{\langle V\rangle_N}{V}\right) \\ &= P(V)_{\mathrm{mw}}\left(1+\frac{\langle V\rangle_N}{V}\right) \\ &= P(V)_{\mathrm{mw}}\left\{1+\left(\frac{2V}{3c}\right)\frac{A[1-(2e^{-A}-1)^N]}{e^A-1}\right\}. \end{aligned} \tag{24}$$

Any $\langle Q\rangle_N$, e.g., $\langle V\rangle_N$, $\langle f\rangle_{N+\frac{1}{2}}$, or $\langle P^*\rangle_{N+\frac{1}{2}}$, is defined for a given $\pm|V|$ *pair*, i.e., for a given $|V|$ — it is *un*defined and can*not* even be calculated given *only* a *single* value of V, e.g., given *only* $+|V|$ *alone* or given *only* $-|V|$ *alone*. $\langle Q\rangle_N$ can be written in the more detailed form $\langle Q(|V|)\rangle_N$; by contrast, the expression $\langle Q(V)\rangle_N$ is meaningless. Since (20) and (24) are correct to first order in V/c, by (24), $P(|V|)_N = P(+|V|)_N + P(-|V|)_N = P(|V|)_0 = P(|V|)_{\mathrm{mw}}$ to first order in $|V|/c$; hence, to first order in $|V|/c$, any average $\langle\langle Q\rangle\rangle_N$ over $P(|V|)_{\mathrm{mw}}$ equals that over $P(|V|)_N$ itself. [Of course, *initially*, at $N=0$, $P(|V|)_N = P(|V|)_0 = P(|V|)_{\mathrm{mw}}$ *exactly*, not merely to first order in $|V|/c$; hence, any average $\langle\langle Q\rangle\rangle_0$ over $P(|V|)_{\mathrm{mw}}$ is *exact*.] The following five averages over $P(|V|)_{\mathrm{mw}}$ will be useful: $\langle\langle|V|\rangle\rangle_{\mathrm{mw}} = (2kT/\pi M)^{1/2}$, $\langle\langle V^2\rangle\rangle_{\mathrm{mw}} = kT/M$, $\langle\langle|V|^3\rangle\rangle_{\mathrm{mw}} = [2(kT/M)^3]^{1/2}$, $\langle\langle V^4\rangle\rangle_{\mathrm{mw}} = 3(kT/M)^2 = 3\langle\langle V^2\rangle\rangle^2_{\mathrm{mw}}$, and $\langle\langle|V|^5\rangle\rangle_{\mathrm{mw}} = 8[2(kT/M)^5]^{1/2}$. [Of course, *numerically*, these five averages are identical whether taken over $P(|V|)_{\mathrm{mw}}$ or over $P(V)_{\mathrm{mw}}$. But, *conceptually*, as per the first two sentences of this paragraph — and anticipating the next paragraph — they are more correctly taken over $P(|V|)_{\mathrm{mw}}$.] Averaging over any *one* given $\pm|V|$ pair to obtain $\langle Q\rangle_N$ first, and subsequently averaging over *all* $|V|$ to obtain $\langle\langle Q\rangle\rangle_N$, is preferable to attempting to obtain $\langle\langle Q\rangle\rangle_N$ directly because, e.g.: (a) the former procedure is easier, (b) *both* $\langle Q\rangle_N$ *and* $\langle\langle Q\rangle\rangle_N$ are thus obtained, and (c) the $|V|$-dependence of $F-R$ is thus accounted for — e.g., as per application of (5) and (6) to the last terms of (11), (17), and (18) in order to obtain (19), (21), and (22), respectively.

Averaging V^2, $|V|^3$, and $|V|^5$ in (19), (21), and (22), respectively, over $P(|V|)_{\mathrm{mw}}$ (as per the immediately preceding paragraph) yields (25), (27), and (28), respectively. So that (25) – (28) are a complete set of equations, we restate the last line of (24) as (26). Thus, we obtain

$$\begin{aligned}\langle\langle V\rangle\rangle_N &= \frac{2\,\langle\langle V^2\rangle\rangle_{\text{mw}}}{3c}\frac{A[1-(2e^{-A}-1)^N]}{e^A-1}\\ &= \frac{2kT}{3Mc}\frac{A[1-(2e^{-A}-1)^N]}{e^A-1},\end{aligned} \tag{25}$$

$$P(V)_N = P(V)_{\text{mw}}\left\{1+\left(\frac{2V}{3c}\right)\frac{A[1-(2e^{-A}-1)^N]}{e^A-1}\right\}, \tag{26}$$

$$\begin{aligned}\langle\langle f\rangle\rangle_{N+\frac{1}{2}} &= \frac{4M\,\langle\langle |V|^3\rangle\rangle_{\text{mw}}}{3Lc}Ae^{-A}(2e^{-A}-1)^N\\ &= \frac{4}{3L}\left[\frac{2(kT)^3}{Mc^2}\right]^{1/2}Ae^{-A}(2e^{-A}-1)^N,\end{aligned} \tag{27}$$

and

$$\begin{aligned}\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}} &= \left(\frac{8M\,\langle\langle |V|^5\rangle\rangle_{\text{mw}}}{9Lc^2}\right)\frac{A^2e^{-A}(2e^{-A}-1)^N[1-e^{-A}(2e^{-A}-1)^N]}{e^A-1}\\ &= \frac{64}{9Lc^2}\left[\frac{2(kT)^5}{M^3}\right]^{1/2}\frac{A^2e^{-A}(2e^{-A}-1)^N[1-e^{-A}(2e^{-A}-1)^N]}{e^A-1}.\end{aligned} \tag{28}$$

(An alternative derivation of $\langle f\rangle_{N+\frac{1}{2}}$ and $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$ is given in Appendix E.)

Note that specification to $P(V)_{\text{mw}} = \frac{1}{2}P(|V|)_{\text{mw}}$ is *not* required for the validity of our analyses: For example, $P(V)_{\text{mw}}$ is *specifically* applied in the last two lines of (24) and in (26); and $P(|V|)_{\text{mw}}$ in (25), (27), and (28). By contrast, for example, (7) – (22) and the first line of (24) are valid not only for $P(V)_0 = P(V)_{\text{mw}} = \frac{1}{2}P(|V|)_{\text{mw}}$, but for *any* $P(V)_0$ — whether continuous, discrete, or possessing both continuous and discrete ranges — that is symmetrical about $V = 0$. {Of course, (1) – (6) are valid for *any* $P(V)$ *whatsoever* [including, e.g., a Dirac δ-function, in which case either (5) or (6) but (unless $V = |V| = 0$) not both would obtain.]} For any *one* given value of $|V|$, i.e., for any *one* given $\pm|V|$ pair, velocity-dependent fluctuations behave identically — and break the randomness of Brownian motion identically — given *any* $P(V)_0$ that is symmetrical about $V = 0$. Note, in particular, as per (12), (20), and the first line of (24), that, for any given V, the bias of $P(V)_N$ from $P(V)_0$ is identical given *any* $P(V)_0$ that is symmetrical about $V = 0$. Considering *all* V, and hence also *all* $|V|$, $P(V)_{\text{mw}} = \frac{1}{2}P(|V|)_{\text{mw}}$ is the symmetrical velocity probability density — indeed, *the* velocity probability density — corresponding to maximum entropy. Hence, $P(V)_{\text{mw}} = \frac{1}{2}P(|V|)_{\text{mw}}$ is employed in this paper — but with the view that generalization is possible to *any* $P(V)_0$ that is symmetrical about $V = 0$.

3. Negentropy production, and formulations of the second law

Positive values (however small) of $\langle V\rangle_N$, $\left|P(V)_N-\frac{1}{2}\right|$, $\langle f\rangle_{N+\frac{1}{2}}$, $\langle P^*\rangle_{N+\frac{1}{2}}$, $\langle\langle V\rangle\rangle_N$, $|P(V)_N - P(V)_{\text{mw}}|$, $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$, and $\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}}$ *despite TEQ* challenge the second law. A positive value of

$\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ *despite TEQ* is our *primary* challenge thereto, because if $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ overcomes a conservative resisting force, equal in magnitude but opposite in direction to $\langle\langle f \rangle\rangle_{N+\frac{1}{2}}$, then there obtains an uncompensated *negative* time rate of change in *total* entropy S:

$$\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}} > 0 \implies dS/dt = -\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}} \Big/ T < 0. \tag{29}$$

Perhaps the simplest such conservative resisting force, $Mg\sin(-\theta) = -Mg\sin\theta \doteq -Mg\theta$, is obtained by sloping our system very slightly upwards towards the $+X$ direction — as per Fig. 1 and the two immediately following paragraphs, very slightly upwards towards the right given a right-handed Cartesian coordinate system, or into a very gentle counterclockwise upward spiral given a right-handed cylindrical coordinate system — at a very small slope angle θ ($0 < \theta \ll 1$ rad; also, $0 < \theta \ll mgL/kT = AL/H_{\text{net}}$), such that $\langle\langle f \rangle\rangle_{N+\frac{1}{2}} = Mg\sin\theta \doteq Mg\theta$. (If, instead, the resisting force is frictional and hence *non*conservative, then it can be overcome at steady state indefinitely — frictional dissipation being *recycled* into power P^* — *despite TEQ*.) Generation — or regeneration via recycling — of power P^* entails spontaneous momentum flow [1] in challenge of the Zhang formulation [1] of the second law (and hence, as per the last two paragraphs of this Sect. 3, of *all* formulations thereof).

Of course, (29) is true for *all* DP power outputs, e.g., (29) is also true for the DP's Carnot-engine [17] power outputs (reviewed in Appendix F).

But, in view of recent work concerning limitations of validity of certain formulations — especially, of *entropy*-based formulations — of the second law in the quantum regime [6s–6ff], the employment of the *entropy*-based (29) requires justification.

In the classical regime, (a) the Zhang [1] formulation of the second law (no spontaneous momentum flow in an isolated system $\implies$ no systematic motion — most generally, no systematic process — at TEQ), and (b) Thomson's formulation thereof (no extractable work at TEQ), are equivalent to (c) the formulation thereof stating that total entropy (total negentropy) can never decrease (increase), and, indeed, to (d) all other formulations of the second law. But, in the quantum regime, entropy (or, equivalently, negentropy — and hence free energy) is a difficult, non-uniquely-defined concept — as opposed to heat, and especially to work [6s–6ff]. Hence, in the quantum regime, (a) and (b) immediately above are preferable to (c) [and (d)] immediately above. This present paper deals only with the classical regime — except for the last four paragraphs of this Sect. 3, a digression concerning limited aspects of the quantum regime in Sect. 6, and a few very brief mentions elsewhere. This present paper is based primarily on (a) immediately above — which implies (b) immediately above *always*, and, apart from difficulties in the quantum regime [6s–6ff], also (c) [and (d)] immediately above. Nevertheless: Insofar as this present paper is concerned, certainly outside of Sect. 6 — and, owing to the *limited* nature of said quantum aspects, probably even in Sect. 6 — (c) immediately above [which justifies the employment of *entropy* in (29)], and also (d) immediately above, still retain validity. [As an aside, note that the usual statement of (b) immediately above — no extractable work *via cyclic processes* at TEQ — is too restrictive. If a system is capable of doing work *even only on a one-time basis via a non*cyclic process — e.g., via a *one-time* isothermal expansion of a gas initially constrained to within less than the total volume of its container — then it is *not initially* at TEQ: it is at TEQ only *after* the gas has expanded to occupy the total volume of its container and hence is no longer capable of doing work. Thus, deleting "via cyclic processes" yields a more general statement as per (b) immediately above, and in accordance with the first two paragraphs of Sect. 1.]

In *both* the classical *and* the quantum regimes — but primarily in the quantum regime (particularly for *finite* quantum systems [6ee]), wherein different formulations of the second law can be *in*equivalent

[6s–6ff]: The Zhang [1] formulation of the second law [(a) immediately above)] is a *maximally* strong formulation thereof, i.e., as strong a formulation thereof as is possible {some other formulation(s), e.g., the Thomson formulation [(b) immediately above], may be equally strong [6s–6dd], but no other formulation can be stronger}: Thus: A challenge to the Zhang [1] formulation of the second law is a challenge to *all* formulations [6s–6ff] thereof — and hence a challenge to *the* second law [6dd]. By contrast (particularly in the quantum regime [6s–6ff]), a challenge to any other formulation(s) [6s–6ff] of the second law (i) may or (ii) may not be a challenge to the Zhang [1] formulation thereof, and hence to *all* formulations, thereof — and hence may be a challenge, respectively, (i) to *the* second law [6dd] or (ii) merely to *a* second law [6dd]. And a *true* challenge must be to *the* — not merely to *a* — second law.

There has recently been discovered a *classical* situation [6gg] wherein the minimal-work-principle formulation of the second law can be *in*valid. [The minimal-work-principle formulation of the second law has previously been investigated in the quantum regime (where it also can be *in*valid) [6v,6w].] But this is not applicable insofar as this present paper is concerned, and in any case does not alter the *maximally strong status* of the Zhang [1] formulation of the second law.

4. Details of Markovian time evolution, and maximization of challenges to the second law

Time evolution is complete at $N = 1$ if $F + R - 1 = 0 \implies A = \ln 2$. This corresponds to an *overall* probability (considering *both* Forward *and* Reverse DP Brownian motion) of $\frac{1}{2}$ (with error only to first order in $|V|/c$ for all $N \geq 1$ and with*out* error at $N = 0$) that any given pawl-peg interaction is either a jump or a bounce, i.e., to $P(> |F + R - 1 = 0 \implies A = \ln 2)_N = P(< |F + R - 1 = 0 \implies A = \ln 2)_N = \frac{1}{2}$ (with error only to first order in $|V|/c$ for all $N \geq 1$ and with*out* error at $N = 0$). (See Appendix A.) As $F + R - 1 \longrightarrow 1 \implies A \longrightarrow 0$, pawl-peg bounces become ever rarer, and hence time evolution becomes ever slower. As $F + R - 1 \longrightarrow -1 \implies A \longrightarrow \infty$, pawl-over-peg jumps become ever rarer, and hence time evolution becomes ever slower.

Time evolution of $\langle V \rangle_N$ and $P(V)_N - \frac{1}{2}$, and likewise of $\langle\langle V \rangle\rangle_N$ and $P(V)_N - P(V)_{\mathrm{mw}}$, towards final steady-state values as $N \longrightarrow \infty$ is monotonic and asymptotic if $0 < F + R - 1 < 1 \implies \ln 2 > A > 0$, diminishing-oscillatory if $-1 < F + R - 1 < 0 \implies \infty > A > \ln 2$, and complete at $N = 1$ if $F + R - 1 = 0 \implies A = \ln 2$.

For general F and R that [as per the sentence immediately following (9d)] are *at most* functions of $|V|$ *only*, and hence *constant* for any *one* given $|V|$ — not merely for the *specific* F and R given by the rightmost terms of (5) and (6), respectively — the functional form of any $\langle Q \rangle_N$ with respect to F, R, and N (and hence with respect to A and N) is *in*dependent of $|V|$. Thus, the values of F, R, and N (and hence of A and N) yielding maximization of any $\langle Q \rangle_N$ are also *in*dependent of $|V|$ — and thus likewise also yield maximization of the corresponding $\langle\langle Q \rangle\rangle_N$.

By inspection of (10) – (14), (19), (20), (25), and (26), $\langle V \rangle_N$ and $\left|P(V)_N - \frac{1}{2}\right|$, and likewise $\langle\langle V \rangle\rangle_N$ and $|P(V)_N - P(V)_{\mathrm{mw}}|$, are maximized by maximizing $(F - R)[1 - (F + R - 1)^N]/(2 - F - R) \implies$ maximizing $A[1-(2e^{-A}-1)^N]/(e^A-1)$: maximization obtains given $0 < A \ll 1$ — implying maximization of $(1 - \frac{1}{2}A)[1 - (1 - 2A)^N]$ — which is maximized at unity by letting $A \longrightarrow 0$ and $N \longrightarrow \infty$, but with $N \longrightarrow \infty$ sufficiently faster than $A \longrightarrow 0$ such that $(1 - 2A)^N \longrightarrow 0$. We thus obtain the *absolute* maxima

$$\begin{aligned} &\langle V \rangle_{N,\max} = \langle V \rangle_\infty |(A \longrightarrow 0) = 2V^2/3c \\ &\implies \langle\langle V \rangle\rangle_{N,\max} = \langle\langle V \rangle\rangle_\infty |(A \longrightarrow 0) = 2\left\langle\left\langle V^2 \right\rangle\right\rangle_{\mathrm{mw}} \big/ 3c = 2kT/3Mc \end{aligned} \tag{30}$$

and

$$\left|P(V)_N - \frac{1}{2}\right|_{\max} \Big/ \frac{1}{2} = \left|P(V|A \longrightarrow 0)_\infty - \frac{1}{2}\right| \Big/ \frac{1}{2}$$
$$= |P(V)_N - P(V)_{\mathrm{mw}}|_{\max} / P(V)_{\mathrm{mw}}$$
$$= |P(V|A \longrightarrow 0)_\infty - P(V)_{\mathrm{mw}}| \, P(V)_{\mathrm{mw}}$$
$$= 2\,|V|\,/3c. \tag{31}$$

If $F + R - 1 \longrightarrow 1 \implies A \longrightarrow 0$, then time evolution becomes infinitely slow — requiring $N \longrightarrow \infty$ — because then pawl-peg bounces become infinitely rare. But any "practical" time evolution is limited to at most a *large but finite* number N of pawl-peg interactions. Hence, $\langle V \rangle_N$ and $\left|P(V)_N - \frac{1}{2}\right|$, and likewise $\langle\langle V \rangle\rangle_N$ and $|P(V)_N - P(V)_{\mathrm{mw}}|$, attain "practical" maxima — corresponding to *small but not infinitesimal* $1-(F+R-1) \implies$ *small but not infinitesimal* A and to *large but not infinite* N — that are almost but not quite as large as the *absolute* maxima given in (30) and (31) corresponding to $F+R-1 \longrightarrow 1 \implies A \longrightarrow 0$ and to $N \longrightarrow \infty$. [This is especially true because, if pawl-peg bounces are *extremely* rare, then the DP has sufficient time *between* pawl-peg bounces so that its X-directional momentum exchanges with the EBR are no longer (as is assumed in our analyses) negligible compared with its X-directional momentum exchanges *at* pawl-peg bounces [12]. This point is further discussed in Appendixes B and C.]

For *small* $N \geq 1$, maximizing $\langle V \rangle_N$ and $\langle\langle V \rangle\rangle_N$ with respect to A by setting $\partial \langle V \rangle_N / \partial A = 0 \implies \partial \langle\langle V \rangle\rangle_N / \partial A = 0$ yields maxima at *moderate* A, because *small* $N \geq 1$ implies only *one or a few* pawl-peg interactions — *not* the *many* pawl-peg interactions that would be required to compensate (or overcompensate) for the *small* probability of pawl-peg bounces corresponding to *small* A. For example, $\langle V \rangle_1$ and $\langle\langle V \rangle\rangle_1$ are maximized at $A = 1$, with $\langle V \rangle_{1,\max} = \langle V \rangle_1 |(A = 1) = 4V^2/3ec \implies \langle\langle V \rangle\rangle_{1,\max} = \langle\langle V \rangle\rangle_1 |(A = 1) = 4 \langle\langle V^2 \rangle\rangle_{\mathrm{mw}} / 3ec = 4kT/3Mec$. Note that these maxima lie in the diminishing-oscillatory regime, as per the second paragraph of this Sect. 4, and hence are larger [by a factor of $2(1 - e^{-1})$] than $\langle V \rangle_\infty |(A = 1) = 2V^2/[3c(e-1)] \implies \langle\langle V \rangle\rangle_\infty |(A = 1) = 2 \langle\langle V^2 \rangle\rangle_{\mathrm{mw}} / [3c(e-1)] = 2kT/[3Mc(e-1)]$, [obtained by putting $N = \infty$ and $A = 1$ into (19) and (25)] — but smaller (by a factor of $2/e$) than the *absolute* maxima as per (30). [Similar results obtain for $\left|P(V)_1 - \frac{1}{2}\right| / \frac{1}{2} = |P(V)_1 - P(V)_{\mathrm{mw}}| / P(V)_{\mathrm{mw}}$, whose maximization at $A = 1$ yields $\left|P(V)_1 - \frac{1}{2}\right|_{\max} / \frac{1}{2} = |P(V)_1 - P(V)_{\mathrm{mw}}|_{\max} / P(V)_{\mathrm{mw}} = \left|P(V|A = 1)_1 - \frac{1}{2}\right| / \frac{1}{2} = |P(V|A = 1)_1 - P(V)_{\mathrm{mw}}| P(V)_{\mathrm{mw}} = 4\,|V|\,/3ec$.] For another example, $\langle V \rangle_2$ and $\langle\langle V \rangle\rangle_2$ are maximized at $A = \frac{1}{2}$ (< 1 as expected), but at the same values as $\langle V \rangle_1$ and $\langle\langle V \rangle\rangle_1$, respectively; and similarly for $\left|P(V)_2 - \frac{1}{2}\right| / \frac{1}{2} = |P(V)_2 - P(V)_{\mathrm{mw}}| / P(V)_{\mathrm{mw}}$.

By contrast, via inspection of (17), (21), and (27), $\langle f \rangle_{N+\frac{1}{2}}$ ($\langle\langle f \rangle\rangle_{N+\frac{1}{2}}$) are maximized by: (a) first, setting $\langle f \rangle_{N+\frac{1}{2}} = \langle f \rangle_{\frac{1}{2}}$ ($\langle\langle f \rangle\rangle_{N+\frac{1}{2}} = \langle\langle f \rangle\rangle_{\frac{1}{2}}$), thereby maximizing $(F + R - 1)^N$ at unity $\implies$ maximizing $(2e^{-A} - 1)^N$ at unity; and (b) then, setting $d(Ae^{-A})/dA = 0 \implies A = 1 \implies (Ae^{-A})_{\max} = e^{-1}$. We thus obtain the *absolute* maxima

$$\langle f \rangle_{N+\frac{1}{2},\max} = \langle f \rangle_{\frac{1}{2}} \Big| (A = 1) = 4M\,|V|^3\,/3ecL$$
$$\implies \langle\langle f \rangle\rangle_{N+\frac{1}{2},\max} = \langle\langle f \rangle\rangle_{\frac{1}{2}} \Big| (A = 1) = 4M \left\langle\left\langle |V|^3 \right\rangle\right\rangle_{\mathrm{mw}} / 3ecL = \frac{4[2(kT)^3/M]^{1/2}}{3ecL}. \tag{32}$$

Time evolution of $\langle f \rangle_{N+\frac{1}{2}}$ and $\langle\langle f \rangle\rangle_{N+\frac{1}{2}}$ towards 0 as $N \longrightarrow \infty$ is monotonic and asymptotic through positive values for all $N \geq 0$ if $0 < F + R - 1 < 1 \implies \ln 2 > A > 0$, diminishing-oscillatory through positive (negative) values at all even (odd) $N \geq 0$ if $-1 < F + R - 1 < 0 \implies \infty > A > \ln 2$, and complete at $N = 1$ if $F + R - 1 = 0 \implies A = \ln 2$.

Thus, as per (18) and (28), in order to maximize $\langle P^* \rangle_{N+\frac{1}{2}}$ ($\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$) and hence, by Sect. 3, $-dS/dt$ — our *primary* challenge to the second law — we should *not* allow the Markovian time evolution of our DP to approach as closely as is "practical" (as per the fifth paragraph of this Sect. 4 and as per Appendix B) to its *final* steady state $N \longrightarrow \infty$. Rather, we should allow this time evolution to proceed *only* to $N_{\text{opt}} + 1$, where N_{opt} is the *optimum finite* value of N; also set A at its corresponding *optimum finite* value A_{opt} (*not* $A \longrightarrow 0$); and then let the DP do work against a conservative resisting force equal in magnitude but opposite in direction to $\langle f \rangle_{N_{\text{opt}}+\frac{1}{2}}$ ($\langle\langle f \rangle\rangle_{N_{\text{opt}}+\frac{1}{2}}$) in this *imposed* steady state: N_{opt} and A_{opt} will be derived shortly. {If, instead, the resisting force is *non*conservative, e.g., friction, then $\langle P^* \rangle_{N+\frac{1}{2}}$ and $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ are still thereby maximized, even though $-dS/dt$ is then dissipated as fast as it is (re)generated via recycling [recall the last two sentences of the paragraph containing (29)].} In special cases, particular optimizations may correspond to the immediately aforementioned general ones, e.g., setting $\theta = \theta_{\text{opt}}$ (with $\theta_{\text{opt}} > 0$ and finite) if $\langle\langle f \rangle\rangle_{N+\frac{1}{2}} = Mg \sin\theta \doteq Mg\theta$ as per the sentence immediately following (29).

As an aside, note that the DP's Carnot-engine [17] power outputs (reviewed in Appendix F), *are* maximized (at $\theta = 0$) if the Markovian time evolution of our DP *is* allowed to approach as closely as is "practical" (as per the fifth paragraph of this Sect. 4 and as per Appendix B) to its *final* steady state, with $A \longrightarrow 0$ and $N \longrightarrow \infty$ as per the first sentence of the paragraph containing (30) and (31) [17], corresponding (as closely as "practical") to the *absolute* maxima given by (30) and (31).

Maximizing $\langle P^* \rangle_{N+\frac{1}{2}}$ and $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ with respect to N at given fixed $F + R - 1 \implies$ given fixed A, by setting

$$\partial \langle P^* \rangle_{N+\frac{1}{2}} \Big/ \partial N = 0 \implies \partial \langle\langle P^* \rangle\rangle_{N+\frac{1}{2}} \Big/ \partial N = 0, \tag{33}$$

yields, for the optimum value of N,

$$N_{\text{opt}} = -\ln(F + R)/\ln(F + R - 1) \implies N_{\text{opt}} = -\ln(2e^{-A})/\ln(2e^{-A} - 1). \tag{34}$$

Obviously, (34) is valid only if $0 \leq F + R - 1 < 1 \implies \ln 2 \geq A > 0$. Also, obviously, if (34) yields a non-whole-number value for N_{opt}, then the *actual* value of N_{opt} equals the whole-number value either immediately smaller or immediately larger than the non-whole-number value yielded by (34). If $-1 < F + R - 1 \leq 0 \implies \infty > A \geq \ln 2$, and also if $0 \leq F + R - 1 < 1 \implies \ln 2 \geq A > 0$ but with $F + R - 1$ sufficiently close to $0 \implies A$ sufficiently close to $\ln 2$ such that (34) yields N_{opt} sufficiently close to 0 as opposed to 1, then $N_{\text{opt}} = 0$.

By (34) and the three immediately following sentences: As $F + R - 1$ increases from 0 to $1 \implies A$ decreases from $\ln 2$ to 0, N_{opt} increases monotonically from 0 to ∞. (By the fifth paragraph of this Sect. 4 and by Appendix B, infinitesimally small $1 - (F + R - 1) \implies$ infinitesimally small A and infinitely large N are not "practical".) By (18), (22), and (28), the immediately preceding paragraph, and anticipating the two immediately following paragraphs: (a) If $F + R - 1 = 0 \implies A = \ln 2$, then $\langle P^* \rangle_{\frac{1}{2}} > 0$, $\langle\langle P^* \rangle\rangle_{\frac{1}{2}} > 0$, and, for all $N \geq 1$, $\langle P^* \rangle_{N+\frac{1}{2}} = \langle\langle P^* \rangle\rangle_{N+\frac{1}{2}} = 0$. (b) If $-1 < F + R - 1 < 0 \implies \infty > A > \ln 2$, then $\langle P^* \rangle_{N+\frac{1}{2}} \longrightarrow 0$ and $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}} \longrightarrow 0$ via ever-diminishing oscillations as $N \longrightarrow \infty$, being positive (negative) at all even (odd) $N \geq 0$. (c) If $0 < F + R - 1 < 1 \implies \ln 2 > A > 0$, then $\langle P^* \rangle_{N+\frac{1}{2}}$ and $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ may reach their maxima at any $N \geq 0$.

Only for $N = 0$ can $\langle P^* \rangle_{N+\frac{1}{2}}$ and $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ be maximized analytically with respect to A at given fixed N by setting

$$\partial \langle P^* \rangle_{N+\frac{1}{2}} \Big/ \partial A = 0 \implies \partial \langle\langle P^* \rangle\rangle_{N+\frac{1}{2}} \Big/ \partial A = 0. \tag{35}$$

For all $N \geq 1$, (35) must be solved numerically. [We neglect the trivial analytical solution of (35), which yields $A = 0$ and corresponds to $\langle P^* \rangle_{N+\frac{1}{2}} = \langle\langle P^* \rangle\rangle_{N+\frac{1}{2}} = 0$ for all $N \geq 0$.] Solving (35) analytically for $N = 0$ yields, for the corresponding optimum value of A,

$$A_{\text{opt}}|(N = 0) = 1, \tag{36}$$

and, for the corresponding maximum values of $\langle P^* \rangle_{\frac{1}{2}}$ and $\langle\langle P^* \rangle\rangle_{\frac{1}{2}}$,

$$\langle P^* \rangle_{\frac{1}{2},\max} = \langle P^* \rangle_{\frac{1}{2}} \Big| (A = 1) = 8M |V|^5 / [(3ec)^2 L] \tag{37}$$

and

$$\langle\langle P^* \rangle\rangle_{\frac{1}{2},\max} = \langle\langle P^* \rangle\rangle_{\frac{1}{2}} \Big| (A = 1) = \frac{8M \langle\langle |V|^5 \rangle\rangle_{\text{mw}}}{(3ec)^2 L} = \frac{64[2(kT)^5/M^3)]^{1/2}}{(3ec)^2 L}, \tag{38}$$

respectively. Note that (36) is consistent with the third sentence following (34). Equal and/or higher maxima — if any exist — of $\langle P^* \rangle_{N+\frac{1}{2}}$ and $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ for $N \geq 1$ [corresponding to $A_{\text{opt}}(N)$ in the range to be given shortly by (39)], can be found numerically.

By (34) and the three immediately following sentences, corresponding to *all* maxima of $\langle P^* \rangle_{N+\frac{1}{2}}$ [$\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$] for *all* $N \geq 1$ {whether or not any of these maxima equal or exceed $\langle P^* \rangle_{\frac{1}{2},\max}$ of (37) [$\langle\langle P^* \rangle\rangle_{\frac{1}{2},\max}$ of (38)]},

$$0 < A_{\text{opt}}|(N \geq 1) \leq [A_{\text{opt}}|(N = 1)]_{\max} < \ln 2, \tag{39}$$

where $[A_{\text{opt}}|(N = 1)]_{\max}$ is, as per the third sentence following (34), the *maximum* value of $A_{\text{opt}}(N)$ that corresponds to $N = 1$ rather than to $N = 0$. $A_{\text{opt}}(N)$ decreases monotonically and asymptotically towards 0 as $N \longrightarrow \infty$. (But note the "practical" limits as per the fifth paragraph of this Sect. 4 and as per Appendix B.)

We conclude this Sect. 4 by considering [assuming, for simplicity, the *specific* F and R given by the rightmost terms of (5) and (6), respectively], the quantity

$$\begin{aligned} &F - R = 4Ae^{-A} |V| / 3c \\ &\implies \langle\langle F - R \rangle\rangle_{\text{mw}} = 4Ae^{-A} \langle\langle |V| \rangle\rangle_{\text{mw}} 3c = 4Ae^{-A}(2kT/\pi M)^{1/2}/3c, \end{aligned} \tag{40}$$

an important measure of the degree to which the randomness of our DP's Brownian motion is broken — and maximization thereof. [Considering any *one* given $\pm |V|$ pair, $F - R$ does not involve an average, and hence is not enclosed within single angular brackets in (40) and (41).] Maximizing $F - R$ and $\langle\langle F - R \rangle\rangle_{\text{mw}}$ with respect to A, i.e., setting $d(Ae^{-A})/dA = 0 \implies A = 1 \implies (Ae^{-A})_{\max} = e^{-1}$, yields

$$\begin{aligned} &(F - R)_{\max} = 4 |V| / 3ec \\ &\implies \langle\langle F - R \rangle\rangle_{\text{mw,max}} = 4 \langle\langle |V| \rangle\rangle_{\text{mw}} / 3ec = 4(2kT/\pi M)^{1/2}/3ec. \end{aligned} \tag{41}$$

Given (41), *all* other measures of the degree to which the randomness of our DP's Brownian motion is broken are also maximized immediately following the *first* step of time evolution, i.e., *all* other $\langle Q\rangle_N$ and $\langle\langle Q\rangle\rangle_N$ are also maximized at $N = 1$ (if, e.g., $Q = f$ or $Q = P^*$, at the $N = 0 \longrightarrow N = 1$ transition, i.e., at the $1\,\mathrm{st}$ pawl-peg interaction) and at $A = 1$. If $Q = f$, and possibly if $Q = P^*$, these maxima are absolute, i.e., not equaled or exceeded for any transition ending at $N > 1$ — in contrast with maximization if, e.g., $Q = V$. For $N > 1$ (and for transitions ending at $N > 1$), (41) does not, in general, correspond to maximization of $\langle Q\rangle_N$ and $\langle\langle Q\rangle\rangle_N$ (whether absolute or merely for the given N). [Note that, in contrast with (41) — which corresponds to maximization at $N = 1$ (or at the $N = 0 \longrightarrow N = 1$ transition, i.e., at the $1\,\mathrm{st}$ pawl-peg interaction) given $A = 1$ — completion of time evolution at $N = 1$ corresponds to $F + R - 1 = 0 \implies A = \ln 2$.]

5. Scaling

Assuming uniform scaling and the validity of (28) and (38), DP size $\propto L$ and $M \propto L^3$, so $\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}} \propto L^{-11/2}$; and power density $\propto \langle\langle P^*\rangle\rangle_{N+\frac{1}{2}} \Big/ L^3 \propto \langle\langle P^*\rangle\rangle_{N+\frac{1}{2}} \Big/ M \propto L^{-17/2}$. Thus, $\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}}$ is maximized by minimizing system size, and $\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}} \Big/ L^3 \propto \langle\langle P^*\rangle\rangle_{N+\frac{1}{2}} \Big/ M$ is maximized even more strongly by both minimizing system size and maximizing the number of systems operating in parallel [18]. Also, both power and power density scale as $T^{5/2}$. As per (29), maximizing power (power density) also maximizes the time rate of the associated total negentropy production (total negentropy production density).

In correction of a previous error [19], $|T_+(V) - T_-(V)| = 4T\,|V|\,/3c$, the magnitude of the temperature difference between the $+X$ and $-X$ disk faces corresponding to $V = \pm\,|V|$ [as per (2), the two immediately following sentences, and the paragraph immediately thereafter], can*not* be reduced via *diffraction* of EBR around the disk, not even if the disk's diameter and thickness are small (linear dimensions $\lesssim hc/kT$) or even very small (linear dimensions $\ll hc/kT$) compared with the wavelength of a typical EBR photon at temperature T ($\approx hc/kT$) [20]. An EBR photon approaching the disk from, e.g., the $+X$ direction can*not*, say, be diffracted into a "U-turn" path, thence impinging on the disk from the $-X$ direction: this requires (forbidden) backwards propagation of Huygens' wavelets [20b,20c] and violates conservation of momentum [20c]! (Diffraction can, of course, occur "around a corner", but *not* into a U-turn path [20b,20c].) *Diffraction* is thereby forbidden from reducing the opacity of a small (linear dimensions $\lesssim hc/kT$) or even of a very small (linear dimensions $\ll hc/kT$) disk and hence from degrading $|T_+(V) - T_-(V)|$ [20b,20c].

Nevertheless, *diffraction* aside, a small (linear dimensions $\lesssim hc/kT$) disk, and especially a very small (linear dimensions $\ll hc/kT$) disk, typically *does* suffer from small opacity. For a typical very small (linear dimensions $\ll hc/kT$) disk, the efficiency of absorption/(re)radiation of EBR *per unit of DP volume* (and hence, assuming uniform scaling, also *per unit of DP mass*) is *in*dependent of DP size [21] — thereby rendering a very thin (thickness $\ll hc/kT$) disk highly transparent. Said transparency degrades DP performance by (a) reduced probability of absorption/(re)radiation of any given EBR photon, and (b) rendering the pawl almost as likely to be impinged on by an EBR photon that *is* absorbed/(re)radiated emanating from the $-X$ direction as by one emanating from the $+X$ direction. Hence, typically, for such a pawl, $T_+(V) = T\left(1 + \frac{2\gamma V}{3c}\right)$, where $0 < \gamma \ll 1$, i.e., $|T_+(V) - T| = \frac{1}{2}\,|T_+(V) - T_-(V)|$ *is* seriously degraded in comparison with (2) — and thus DP performance *is also* seriously degraded. Since small DP size (and mass) with*out* appreciable degradation of DP performance is necessary for significant — or even measurable —

power and negentropy production densities, the question arises as to whether or not said degradation in small (linear dimensions $\lesssim hc/kT$), and even very small (linear dimensions $\ll hc/kT$), DPs can be overcome.

Perhaps, it can be overcome via a DP possessing one or more of the following *un*typical properties: (a) Overlapping resonances: If the DP is comprised of atoms and/or molecules whose resonances overlap to significantly "cover" the Planck spectrum corresponding to T, then the DP might be highly opaque even if it is very small (linear dimensions $\ll hc/kT$) [22]. (b) An internal reflective shield: A nonreflective [purely absorptive/(re)radiative] nonresonant material cannot be both thin ($\lesssim hc/kT$) and opaque to EBR corresponding to T [23]. But a reflective material (even if *non*resonant) can be [24]. Therefore, a thin reflective (if *also* resonant, so much the better) midsection comprising the "center slab" of the disk separates its absorptive/(re)radiative $+X$ and $-X$ faces not only spatially, but — more importantly — thermally. [Of course, whether or not such a reflective shield is present, the $+X$ and $-X$ disk faces *themselves* must be absorptive/(re)radiative — not reflective: any purely reflective material obviously can never (re)thermalize!] (c) Alternatively (see Appendixes G and H), perhaps a nonrelativistic positive-rest-mass thermal background medium at temperature T might be made preponderant over the EBR [25], in which case $c \longrightarrow |U|$, with $|U|$ on the order of a typical thermal or sonic molecular speed in said medium, rather than the speed of light in vacuum — which yields the advantage, for any given DP size and mass, of $|U| \ll c \implies |V| / |U| \gg |V| /c$ [25]. A further advantage obtains if DP size and mass given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR can be smaller than those given the EBR being the *sole* thermal background medium. (Even given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR, there seems to be no advantage in "excluding" the EBR corresponding to T: such "exclusion" begins to obtain if the DP is enclosed within, say, a conducting shell of diameter $\lesssim hc/kT \approx$ wavelength of typical EBR photon at temperature T, and obtains *strongly* if said diameter $\ll hc/kT$.)

6. A digression concerning limited aspects of DP operation in the quantum regime

For brevity in notation in this Sect. 6, we first define, in the *classical* regime,

$$F \equiv \frac{1}{2}(\mu + \epsilon) \tag{42}$$

and

$$R \equiv \frac{1}{2}(\mu - \epsilon). \tag{43}$$

Note that (42) and (43) imply, and are implied by,

$$\mu \equiv F + R \tag{44}$$

and

$$\epsilon \equiv F - R. \tag{45}$$

Also, for simplicity, in this Sect. 6, we consider any *one* given $\pm |V|$ pair, i.e., any *one* given $|V|$; except for the last paragraph thereof, wherein averages over *all* $\pm |V|$ pairs, i.e., over *all* $|V|$, are briefly mentioned.

Applying (44) and (45) in the *classical* regime, we can rewrite the last line of (10), (11), (15), (16), (17), and (18), respectively, as

$$P(\pm)_N = \frac{1}{2}\left\{1 \pm \frac{\epsilon[1-(\mu-1)^N]}{2-\mu}\right\}, \tag{46}$$

$$\langle V\rangle_N = |V|\,\frac{\epsilon[1-(\mu-1)^N]}{2-\mu}, \tag{47}$$

$$\langle V\rangle_{N+\frac{1}{2}} = |V|\,\frac{\epsilon[2-\mu(\mu-1)^N]}{2(2-\mu)}, \tag{48}$$

$$\langle \Delta V\rangle_{N+\frac{1}{2}} = |V|\,\epsilon(\mu-1)^N, \tag{49}$$

$$\langle f\rangle_{N+\frac{1}{2}} = \left(\frac{MV^2}{L}\right)\epsilon(\mu-1)^N, \tag{50}$$

and

$$\langle P^*\rangle_{N+\frac{1}{2}} = \left(\frac{M\,|V|^3}{2L}\right)\frac{\epsilon^2(\mu-1)^N[2-\mu(\mu-1)^N]}{2-\mu}. \tag{51}$$

The ϵ-, μ-, and N-functionalities are mutually *in*dependent. At constant μ and N [except for the trivial case $N=0$ in (46) and (47), corresponding to $\left|P(\pm)_0 - \frac{1}{2}\right| = 0$ and to $\langle V\rangle_0 = 0$, respectively], all $\langle Q\rangle_N$, as per (46) – (51), are $\propto \epsilon$ (except $\langle P^*\rangle_{N+\frac{1}{2}} \propto \epsilon^2$).

We now explore limited aspects of DP operation in the *quantum* regime, considering the pawl's quantum-mechanical tunneling through [26a] (or quantum-mechanical — as opposed to classical — jumping over [26b]) pegs when it would classically bounce, and its quantum-mechanical reflection from pegs when it would classically jump [26]. For simplicity, we assume in this Sect. 6 — as we do throughout this paper — (in addition to our nonrelativistic assumptions as per the last sentence of the paragraph immediately following Fig. 1): (a) that $m\,\langle|v|\rangle_{\mathrm{mw}}\,(Z-Z_{\min})_{\mathrm{scale}} = m(2kT/\pi m)^{1/2}(kT/mg) = (2/\pi m)^{1/2}(kT)^{3/2}/g \approx (kT)^{3/2}/m^{1/2}g \gg h$, so that the pawl's Z-directional thermal (Brownian) motion can still be treated classically; (b) that $M\,\langle|V|\rangle_{\mathrm{mw}}\,L = M(2kT/\pi M)^{1/2}L = (2kTM/\pi)^{1/2}L \approx (kTM)^{1/2}L \gg h$, so that the DP's X-directional thermal (Brownian) motion can still be treated classically; and (c) that (anticipating Appendix A) the combined pawl-plus-peg X-directional thickness is $\ll L$, so that quantum-mechanically (as well as classically, as per Appendix A) Δt is only negligibly affected thereby.

Classically, given DP Brownian motion at $V = +|V|$, the pawl's altitude at pawl-peg interaction is $Z > H$ ($Z_{\min} \le Z < H$) with probability F $(1-F)$, corresponding — with *certainty* — to the interaction being a jump (bounce). By contrast, in the *quantum* regime, the pawl can, with *non*zero probability, tunnel through [26a] (or quantum-mechanically — as opposed to classically — jump over [26b]) a peg if $Z_{\min} \le Z < H$, and reflect or bounce from a peg if $Z > H$ [26]. Within approximations (a) and (b) as per the immediately preceding paragraph, quantum-mechanically — as classically — given DP Brownian motion at $V = +|V|$, the pawl's altitude at pawl-peg interaction is $Z > H$ ($Z_{\min} \le Z < H$) with probability F $(1-F)$. But, quantum-mechanically [26]: (i) if $Z_{\min} \le Z < H$, then the pawl will jump with probability $(1-F)\tau_+$ and bounce with probability $(1-F)(1-\tau_+)$, where $\tau(+|V|) \equiv \tau_+$ is the quantum-mechanical probability of tunneling given $V = +|V|$, integrated over the range $Z_{\min} \le Z < H$. And, (ii) if $Z > H$, then the pawl will jump with probability $F(1-\rho_+)$ and bounce with probability $F\rho_+$, where $\rho(+|V|) \equiv \rho_+$ is the quantum-mechanical probability of reflection given $V = +|V|$, integrated over the range $Z > H$. This paragraph obviously also obtains given DP Brownian motion at $V = -|V|$, as per the substitutions

$V = +|V| \longrightarrow V = -|V|$, $F \longrightarrow R$, $\tau(+|V|) \equiv \tau_+ \longrightarrow \tau(-|V|) \equiv \tau_-$, and $\rho(+|V|) \equiv \rho_+ \longrightarrow \rho(-|V|) \equiv \rho_-$. Explicitly, applying (3) – (6) with $H_{\text{net}} \equiv H - Z_{\min} \longrightarrow Z - Z_{\min}$, to first order in $|V|/c$,

$$\tau_\pm \equiv \tau(\pm|V|) = \langle \tau(\pm|V|, Z) \rangle = \frac{\int_{Z_{\min}}^{H} \tau_\pm(Z) P(Z|V = \pm|V|) dZ}{\int_{Z_{\min}}^{H} P(Z|V = \pm|V|) dZ}$$
$$= \frac{\int_{Z_{\min}}^{H} \tau_\pm(Z) \left(1 \pm \frac{2mg(Z-Z_{\min})|V|}{3kTc}\right) e^{-mg(Z-Z_{\min})/kT} dZ}{\int_{Z_{\min}}^{H} \left(1 \pm \frac{2mg(Z-Z_{\min})|V|}{3kTc}\right) e^{-mg(Z-Z_{\min})/kT} dZ} \tag{52}$$

and

$$\rho_\pm \equiv \rho(\pm|V|) = \langle \rho(\pm|V|, Z) \rangle = \frac{\int_{H}^{\infty} \rho_\pm(Z) P(Z|V = \pm|V|) dZ}{\int_{H}^{\infty} P(Z|V = \pm|V|) dZ}$$
$$= \frac{\int_{H}^{\infty} \rho_\pm(Z) \left(1 \pm \frac{2mg(Z-Z_{\min})|V|}{3kTc}\right) e^{-mg(Z-Z_{\min})/kT} dZ}{\int_{H}^{\infty} \left(1 \pm \frac{2mg(Z-Z_{\min})|V|}{3kTc}\right) e^{-mg(Z-Z_{\min})/kT} dZ}. \tag{53}$$

Applying (52) qualitatively [26], for any $|V| > 0$, $\tau_+ > \tau_-$, because, given $Z_{\min} \le Z < H$, Z is [as per (3) – (6) with $H_{\text{net}} \equiv H - Z_{\min} \longrightarrow Z - Z_{\min}$] more probably closer to H if $V = +|V|$ than if $V = -|V|$. Similarly, applying (53) qualitatively [26], for any $|V| > 0$, $\rho_- > \rho_+$, because, given $Z > H$, Z is [as per (3) – (6) with $H_{\text{net}} \equiv H - Z_{\min} \longrightarrow Z - Z_{\min}$] more probably closer to H if $V = -|V|$ than if $V = +|V|$.

Thus, letting the subscript "q" denote "quantum-mechanical", we have {as the quantum-mechanical analog of (5), (9a), and (42) [within said approximations (a) and (b)]}, for the *overall* probability, integrated over *all* $Z \ge Z_{\min}$, given DP Brownian motion at $V = +|V|$, of a pawl-over-peg jump in the $+X$ direction,

$$F_{\text{q}} = F(1 - \rho_+) + (1 - F)\tau_+. \tag{54}$$

Similarly, {as the quantum-mechanical analog of (6), (9b), and (43) [within said approximations (a) and (b)]}, the *overall* probability, integrated over *all* $Z \ge Z_{\min}$, given DP Brownian motion at $V = -|V|$, of a pawl-over-peg jump in the $-X$ direction is

$$R_{\text{q}} = R(1 - \rho_-) + (1 - R)\tau_-. \tag{55}$$

All classical results in this paper *also* apply in the *quantum* regime — within said approximations (a) and (b) — given the substitutions $F \longrightarrow F_{\text{q}}$ and $R \longrightarrow R_{\text{q}}$. Also, note that the correspondence principle is obeyed: *purely* classical behavior is recovered in the limits $\tau_+ \longrightarrow 0$, $\tau_- \longrightarrow 0$, $\rho_- \longrightarrow 0$, and $\rho_+ \longrightarrow 0$.

For still greater brevity in notation, we define, for the remainder of this Sect. 6,

$$\Sigma\tau \equiv \tau_+ + \tau_-, \tag{56}$$

$$\Sigma\rho \equiv \rho_- + \rho_+, \tag{57}$$

$$\Delta\tau \equiv \tau_+ - \tau_- > 0, \tag{58}$$

and

$$\Delta\rho \equiv \rho_- - \rho_+ > 0. \tag{59}$$

The inequalities in (58) and (59) are justified by the two sentences immediately following (53).

Applying (42) – (45) and (54) – (59), we have, as the quantum-mechanical analogs — within said approximations (a) and (b) — of (44) and (45), respectively,

$$\begin{aligned}
\mu_{\rm q} &\equiv F_{\rm q} + R_{\rm q} \\
&= F(1-\rho_+) + (1-F)\tau_+ + R(1-\rho_-) + (1-R)\tau_- \\
&= F + R - F(\rho_+ + \tau_+) - R(\rho_- + \tau_-) + \tau_+ + \tau_- \\
&= \mu - \frac{1}{2}(\mu+\epsilon)(\rho_+ + \tau_+) - \frac{1}{2}(\mu-\epsilon)(\rho_- + \tau_-) + \tau_+ + \tau_- \\
&= \mu - \frac{\mu}{2}(\Sigma\rho + \Sigma\tau) + \frac{\epsilon}{2}(\Delta\rho - \Delta\tau) + \Sigma\tau
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
\epsilon_{\rm q} &\equiv F_{\rm q} - R_{\rm q} \\
&= F(1-\rho_+) + (1-F)\tau_+ - [R(1-\rho_-) + (1-R)\tau_-] \\
&= F - R - F(\rho_+ + \tau_+) + R(\rho_- + \tau_-) + \tau_+ - \tau_- \\
&= \epsilon - \frac{1}{2}(\mu+\epsilon)(\rho_+ + \tau_+) + \frac{1}{2}(\mu-\epsilon)(\rho_- + \tau_-) + \tau_+ - \tau_- \\
&= \epsilon + \frac{\mu}{2}(\Delta\rho - \Delta\tau) - \frac{\epsilon}{2}(\Sigma\rho + \Sigma\tau) + \Delta\tau.
\end{aligned} \tag{61}$$

As per the sentence containing (46) and the two sentences immediately following (55), the *classical* (46) – (51) *also* apply in the *quantum* regime — within said approximations (a) and (b) — given the substitutions $\epsilon \longrightarrow \epsilon_{\rm q}$ and $\mu \longrightarrow \mu_{\rm q}$.

We now consider the simplest nontrivial special case, which obtains at $N = 1$ in (46) and (47), and at the $N = 0 \longrightarrow N = 1$ transition in (48) – (51). [The trivial case is: $N = 0$ in (46) and (47): recall the second sentence following (51).] In this simplest nontrivial special case, ϵ-dependence alone obtains [recall the paragraph immediately following (51)]. Can $\epsilon_{\rm q} > \epsilon$ obtain, i.e., can a quantum DP [within said approximations (a) and (b)] outperform a classical DP in this simplest nontrivial special case? Applying (45) and (61) yields

$$\begin{aligned}
\epsilon_{\rm q} > \epsilon \implies & \frac{\mu}{2}(\Delta\rho - \Delta\tau) - \frac{\epsilon}{2}(\Sigma\rho + \Sigma\tau) + \Delta\tau > 0 \\
\implies & \frac{\epsilon(\Sigma\rho + \Sigma\tau)}{\Delta\tau} + \mu\left(1 - \frac{\Delta\rho}{\Delta\tau}\right) < 2.
\end{aligned} \tag{62}$$

Owing to algebraic difficulty, it is unclear whether or not (62) can be fulfilled for any physically realistic values of quantities appearing therein, let alone whether or not $\epsilon_{\rm q,max} > \epsilon_{\rm max}$.

The second-simplest nontrivial special case — entailing the (mutually *in*dependent) ϵ- and μ-functionalities — obtains in the limit $N \longrightarrow \infty$ (within "practical" limits as per the fifth paragraph of Sect. 4 and as per Appendix B) in (46) – (48). Applying (44), (45), (60), and (61) yields, in this second-simplest nontrivial special case, as the requirement for a quantum DP [within said approximations (a) and (b)] to outperform a classical DP,

$$
\begin{aligned}
&\frac{\epsilon_{\rm q}}{2-\mu_{\rm q}} > \frac{\epsilon}{2-\mu} \\
&\implies \frac{\epsilon + \frac{\mu}{2}(\Delta\rho - \Delta\tau) - \frac{\epsilon}{2}(\Sigma\rho + \Sigma\tau) + \Delta\tau}{2 - [\mu - \frac{\mu}{2}(\Sigma\rho + \Sigma\tau) + \frac{\epsilon}{2}(\Delta\rho - \Delta\tau) + \Sigma\tau]} > \frac{\epsilon}{2-\mu} \\
&\implies (2-\mu)\,[\epsilon + \frac{\mu}{2}(\Delta\rho - \Delta\tau) - \frac{\epsilon}{2}(\Sigma\rho + \Sigma\tau) + \Delta\tau] \\
&> \epsilon\{2 - [\mu - \frac{\mu}{2}(\Sigma\rho + \Sigma\tau) + \frac{\epsilon}{2}(\Delta\rho - \Delta\tau) + \Sigma\tau]\} \\
&\implies \left[\mu - \frac{1}{2}(\mu^2 - \epsilon^2)\right](\Delta\rho - \Delta\tau) + (2-\mu)\Delta\tau > \epsilon\Sigma\rho \\
&\implies \mu\left(1 - \frac{1}{2}\mu\right)(\Delta\rho - \Delta\tau) + (2-\mu)\Delta\tau > \epsilon\Sigma\rho. \qquad (63)
\end{aligned}
$$

The last step of (63) is justified because [given the *specific* F and R as per the rightmost terms of (5) and (6), respectively, and applying (42) – (45)] $\epsilon^2 \ll \mu^2$. It is even less clear — owing to greater algebraic difficulty — whether or not (63) can be fulfilled for any physically realistic values of quantities appearing therein, let alone whether or not $[\epsilon_{\rm q}/(2-\mu_{\rm q})]_{\max} > [\epsilon/(2-\mu)]_{\max}$.

Owing to still greater algebraic difficulty, we will not specifically consider the completely general case — entailing all three (mutually *in*dependent) ϵ-, μ-, and N-functionalities.

By averaging over *all* $\pm|V|$ pairs, i.e., over *all* $|V|$ — similarly as for our classical DP [as per the third paragraph and last four paragraphs of Sect. 2, and in light of the two sentences immediately following (55) and that immediately following (61)] — *overall* quantum DP behavior [within said approximations (a) and (b)] can be similarly derived.

7. Conclusion

In the original classic "Ratchet and Pawl" chapter [4], Feynman's upshot concerning the "Ratchet and Pawl" elucidates the Zhang [1] formulation of the second law:

> "In spite of all our cleverness of lopsided design, if the two temperatures are exactly equal there is no more propensity to turn one way than the other. The moment we look at it, it may be turning one way or the other, but in the long run it gets nowhere. The fact that it gets nowhere is really the fundamental deep principle on which all of thermodynamics is based."

Feynman's ratchet (which manifests *only non*-velocity-dependent fluctuations) can operate — albeit *not* in violation of the second law — *if* negentropy (and hence free energy) is supplied thereto, i.e., at *non*-TEQ; e.g., (a) if the two temperatures are unequal, or (b) if the pawl — assumed to have a sufficiently short thermal response time — is heated (cooled) whenever the wheel turns in the Forward (Reverse) direction.

But Feynman's classic ratchet and pawl manifests asymmetry, i.e., lopsidedness, *only geometrically*, i.e., *only in coordinate space alone*. By contrast, *velocity-dependent* fluctuations — which *spontaneously* break the randomness of our DP's Brownian motion at TEQ — manifest asymmetry, i.e., lopsidedness, in *velocity or momentum space*, hence allowing operation *despite TEQ*: Velocity-dependent fluctuations execute procedure (b) given in the immediately preceding paragraph, but *spontaneously*, i.e., with*out* cost in negentropy (and hence in free energy). Of course, our DP is *also* asymmetrical geometrically: geometrical (coordinate-space) asymmetry plays an auxiliary yet necessary role in the case of our DP — and perhaps in general — to momentum-space-asymmetrical velocity-dependence of fluctuations in the spontaneous breaking of the randomness of Brownian motion at TEQ. But geometrical (coordinate-space) asymmetry *alone* does not *imply* the velocity- or momentum-space asymmetry that seems to be the *central* requirement for this spontaneous randomness-breaking.

If the impedance to Brownian-motional velocity V is thus an asymmetrical function of V *itself*, then can the randomness of V be spontaneously broken, thereby challenging the Zhang [1] formulation — and hence, as per the last four paragraphs of Sect. 3, *all* formulations — of the second law?

The correspondence principle requires that the following necessary — albeit, of course, not sufficient — condition must be satisfied by any *valid* new and/or generalized scientific theory: the new and/or generalized theory must reduce to any more restricted theory that is a special case thereof within the more restricted theory's (narrower) range of validity. Our theory challenging the second law via *velocity-dependence* of fluctuations *is* in accordance with the correspondence principle: it predicts that the second law and all results based thereon remain inviolate if *velocity-dependence* of fluctuations vanishes.

Of course, the five immediately preceding paragraphs do *not* preclude *un*related challenges to the second law that are *not* based on velocity-dependence of fluctuations [5–10]. Also, obviously, the five immediately preceding paragraphs do *not* imply that *every* velocity-dependent effect can spontaneously break the randomness of Brownian motion at TEQ — challenging the second law. For example, it seems unlikely that the Lorentz force — and hence, by extension, *any non*dissipative velocity-dependent force that acts perpendicularly to the velocity *itself* (i.e., to the direction of motion), and hence which can do no work — can thus challenge the second law [27]. [Note: The Zhang formulation of the second law, enunciated in Ref. [1a] (and restated in the first two paragraphs of Sect. 1, with further discussions in the last four paragraphs of Sect. 3, of this present paper), is valid irrespective of whether or not (*non*dissipative) velocity-dependent forces acting perpendicularly to the velocity itself (i.e., to the direction of motion), and hence which can do no work — such as the Lorentz force — can challenge the second law.] Also, of course, (in general, velocity-dependent) *dissipative* frictional forces do *not* challenge the second law — rather, friction (whether velocity-dependent or not) *manifests* the second law.

Questions that have not been addressed or answered in this present paper imply the following: (i) resolving the challenge to the second law (pro or con) posed by our classical velocity-dependent DP model, i.e., posed by our DP per se [as developed initially in Ref. [3k] and (as per Appendix I of this present paper) more quantitatively both in this present paper and in previous shorter versions [28] thereof], and posed by possible classical generalizations of our DP model; (ii) investigating alternative derivations relevant to our DP model; (iii) more thorough study of quantum-mechanical velocity-dependent models; (iv) possible generalization of source(s) of velocity-dependence — in both the classical and quantum regimes — to other than the Doppler effect, if such source(s) exist; (v) investigating whether or not geometrical asymmetry is *always* an auxiliary requirement to velocity-dependence of fluctuations for our challenge to the second law, and the issue of auxiliary requirement(s) in general (in both the classical and quantum regimes); (vi) understanding which manifestation(s) of velocity-dependence can or cannot challenge the second law in both the classical and quantum regimes — recall the last four sentences of the immediately preceding paragraph; and (vii) investigating relationship(s) to other (classical and quantum) challenges to the second law [5–10], including a search for unifying principle(s) behind challenges thereto — whether based on velocity-dependence of fluctuations or not (recall the immediately preceding paragraph).

Perhaps, in passing, it might be noted that there have also been challenges — albeit *un*related to this present paper — to the first [29] and third [30] laws of thermodynamics.

Corrections for Ref. [3k] (Ref. [28b]) are given in Appendix I (Appendix J) of this present paper.

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Appendix A. Separation between consecutive pawl-peg interactions with finite pawl-plus-peg X-directional thickness

Let L_{pc} be the X-directional spatial separation between adjacent peg centers; and let l_{pawl} (l_{peg}) be the X-directional thickness of the pawl (of any peg). The combined pawl-plus-peg X-directional thickness is thus $l_{\mathrm{pawl+peg}} = l_{\mathrm{pawl}} + l_{\mathrm{peg}} \equiv l_+$. In this Appendix A, we denote a pawl-over-peg jump (pawl-peg bounce) by J (B). Let ΔX_{ij} ($i, j = J$ or B) be the X-directional distance traversed by the DP between the midpoints of two consecutive pawl-peg interactions. We first consider any *one* given $\pm |V|$ pair, i.e., any *one* given $|V|$, and subsequently, where indicated, average over *all* $|V|$.

Then, clearly, $\Delta X_{JJ} = L_{\mathrm{pc}}$, $\Delta X_{JB} = \Delta X_{BJ} = L_{\mathrm{pc}} - \frac{1}{2}l_+$, and $\Delta X_{BB} = L_{\mathrm{pc}} - l_+$. The probability $P(J)_N = P(>)_N$ of a jump or $P(B)_N = P(<)_N$ of a bounce at the $N\,\mathrm{th}$ pawl-peg interaction is, strictly,

$$\begin{aligned}
P(J)_N &= P(>)_N = P(>+)_N + P(>-)_N \\
&= P(+)_N P(>|+) + P(-)_N P(>|-) \\
&= FP(+)_N + RP(-)_N \\
&= FP(+)_N + R[1 - P(+)_N] \\
&= R + (F - R)P(+)_N \\
&= \left(1 - \frac{2|V|A}{3c}\right)e^{-A} + \frac{4|V|}{3c}Ae^{-A}P(+)_N \\
\Longrightarrow\ P(B)_N &= P(<)_N = 1 - P(J)_N = 1 - P(>)_N \\
&= 1 - R - (F - R)P(+)_N \\
&= 1 - \left(1 - \frac{2|V|A}{3c}\right)e^{-A} - \frac{4|V|}{3c}Ae^{-A}P(+)_N.
\end{aligned} \tag{A1}$$

The third step of (A1) is justified by (5), (6), (9a), and (9b); the fourth step by $P(+)_N + P(-)_N = 1$; the sixth and last steps by the last steps of (5) and (6); and the seventh step by $P(J)_N + P(B)_N$ $P(>)_N + P(<)_N = 1$.

However, neglecting the N- and V-dependencies results — as per (A1) and the rightmost terms in (5) and (6) — in error only to first order in $|V|/c$ for all $N \geq 1$, and in *no* error given $N = 0 \iff P(+)_N = P(+)_0 = P(-)_N = P(-)_0 = \frac{1}{2}$. Thus, if we instead, to sufficient accuracy, simply take

$$\begin{aligned}
&P(J) = P(J)_0 = P(>)_0 = e^{-A} \\
&\Longrightarrow\ P(B) = P(B)_0 = P(<)_0 = 1 - P(J)_0 = 1 - P(>)_0 = 1 - e^{-A},
\end{aligned} \tag{A2}$$

then the calculations are greatly simplified. Hence, in this Appendix A, we take this approach unless otherwise indicated.

We have

$$\begin{aligned}
\langle \Delta X_{ij} \rangle &= \Delta X_{JJ}[P(J)]^2 + \Delta X_{JB}P(J)P(B) + \Delta X_{BJ}P(B)P(J) + \Delta X_{BB}[P(B)]^2 \\
&= L_{\mathrm{pc}}[P(J)]^2 + 2(L_{\mathrm{pc}} - \frac{1}{2}l_+)P(J)P(B) + (L_{\mathrm{pc}} - l_+)[P(B)]^2 \\
&= \langle\langle X_{ij} \rangle\rangle\,.
\end{aligned} \tag{A3}$$

The first two steps of (A3) entail no approximations. The last step of (A3) is justified because, to said sufficient accuracy, $\langle X_{ij}\rangle$ is *in*dependent of $|V|$.

But (with no approximations)

$$
\begin{aligned}
&P(J)+P(B)=[P(J)+P(B)]^2=[P(J)]^2+[P(B)]^2+2P(J)P(B)=1\\
&\implies 2P(J)P(B)=1-[P(J)]^2-[P(B)]^2.
\end{aligned}
\tag{A4}
$$

Applying (A4) to the second line of (A3), and then applying (A2), yields

$$
\begin{aligned}
\langle\Delta X_{ij}\rangle &= L_{\rm pc}[P(J)]^2+(L_{\rm pc}-\frac{1}{2}l_+)\{1-[P(J)]^2-[P(B)]^2\}+(L_{\rm pc}-l_+)[P(B)]^2\\
&= L_{\rm pc}-\frac{1}{2}l_+\{1-[P(J)]^2+[P(B)]^2\}\\
&= L_{\rm pc}-\frac{1}{2}l_+[1-(e^{-A})^2+(1-e^{-A})^2]\\
&= L_{\rm pc}-\frac{1}{2}l_+(1-e^{-2A}+1-2e^{-A}+e^{-2A})\\
&= L_{\rm pc}-l_+(1-e^{-A})\\
&= \langle\langle X_{ij}\rangle\rangle\,.
\end{aligned}
\tag{A5}
$$

The first two steps of (A5) [as those of (A3)] entail no approximations. Our approximations of neglecting the N- and V-dependencies are applied in the last four steps of (A5). The last step of (A5) [as that of (A3)] is justified because, to said sufficient accuracy, $\langle X_{ij}\rangle$ is *in*dependent of $|V|$.

Thus, if, as we always assume in this paper, $l_+ \ll L_{\rm pc}$, then there is only a negligible difference between choosing, say, $L = L_{\rm pc}$ or $L = \langle\Delta X_{ij}\rangle = \langle\langle\Delta X_{ij}\rangle\rangle$; hence, we can, with only negligible error, take $L = L_{\rm pc}$.

Thus far in this Appendix A, we took ΔX_{ij} ($i,\ j = J$ or B) to be the X-directional distance traversed by the DP between the midpoints of two consecutive pawl-peg interactions. But (with no approximations) $\langle\Delta X_{ij}\rangle = \langle\langle\Delta X_{ij}\rangle\rangle$ would be *identical* if, instead, we took ΔX_{ij} to be said distance traversed by the DP between (a) the beginnings or (b) the ends of two consecutive pawl-peg interactions. For, given either (a) or (b) immediately above, $\Delta X_{JJ} = L_{\rm pc}$, $\Delta X_{BB} = L_{\rm pc} - l_+$; and, clearly, $P(J)$ and $P(B)$ are unchanged. Given (a) immediately above, $\Delta X_{JB} = L_{\rm pc}$, $\Delta X_{BJ} = L_{\rm pc} - l_+$, and $\Delta X_{JB}P(J)P(B) + \Delta X_{BJ}P(B)P(J) = (2L_{\rm pc} - l_+)P(J)P(B) = 2(L_{\rm pc} - \frac{1}{2}l_+)P(J)P(B)$. Given (b) immediately above, $\Delta X_{JB} = L_{\rm pc} - l_+$, $\Delta X_{BJ} = L_{\rm pc}$, and $\Delta X_{JB}P(J)P(B) + \Delta X_{BJ}P(B)P(J) = (2L_{\rm pc} - l_+)P(J)P(B) = 2(L_{\rm pc} - \frac{1}{2}l_+)P(J)P(B)$. In both said cases (a) and (b), *exact* agreement obtains with the first line of (A3), in particular with the sum of the second and third terms on the right-hand side thereof; and with the second line of (A3), in particular with the second term on the right-hand side thereof.

Of course, corresponding to any *one* given $|V|$, the average temporal interval $\langle\Delta t_{ij}\rangle$ separating any two consecutive pawl-peg interactions is simply $\langle\Delta X_{ij}\rangle\,/\,|V|$. Averaging over *all* $\pm|V|$ pairs, i.e., over *all* $|V|$, $\langle\langle\Delta t_{ij}\rangle\rangle = \langle\Delta X_{ij}\rangle\,/\,\langle\langle|V|\rangle\rangle_{\rm mw} = \langle\langle\Delta X_{ij}\rangle\rangle\,/\,\langle\langle|V|\rangle\rangle_{\rm mw} = \langle\Delta X_{ij}\rangle\,(\pi M/2kT)^{1/2} =$
$\langle\langle\Delta X_{ij}\rangle\rangle\,(\pi M/2kT)^{1/2}$.

Appendix B: DP (re)thermalization, minimum "practical" A, and derivation of the requirements for negligibility of EBR drag and rapid DP thermalization

By Sect. 4, A should approach 0 as closely as "practical" (as per the fifth paragraph of Sect. 4) in order to maximize systematic net drift velocity and bias from the Maxwellian velocity probability density — albeit neither for maximizing the force that tends to accelerate the DP in the $+X$ direction, nor for our *primary* objective of maximizing its power output and hence its time rate of negentropy production. A can be very — but not infinitesimally — small, as will now be shown.

First, note that consecutive pawl-peg bounces are — averaging over *both* Forward *and* Reverse DP Brownian motion, and whether considering *one* given $|V|$ or averaging over *all* $|V|$ — spatially separated by $\Delta X_{\text{bounce}} = L/[(1-e^{-A})$. Also, note that $\Delta X_{\text{bounce}} = L/(1-e^{-A})$ should not be confused with ΔX_{BB} (recall Appendix A): ΔX_{BB} is the spatial interval separating consecutive pawl-peg bounces given *zero* intervening pawl-over-peg jumps; by contrast, $\Delta X_{\text{bounce}} = L/(1-e^{-A}) \doteq \Delta X_{BB}/(1-e^{-A})$ is the *average* spatial interval separating consecutive pawl-peg bounces, corresponding to the *average* number $1/(1-e^{-A})-1$ of intervening pawl-over-peg jumps. [Since ΔX_{bounce} — although an average — is, to first order in $|V|/c$, *in*dependent of $|V|$, it is not enclosed within angular brackets. Also, ΔX_{bounce} is not an average over ΔX_{ij} (i, $j = J$ or B) as per Appendix A, especially (A3) and (A5), but, again, rather an average over said number of intervening pawl-over-peg *jumps*.]

Considering *all* $\pm|V|$ pairs, i.e., *all* $|V|$, four time intervals are pertinent: (i) $\langle\langle \Delta t\rangle\rangle = L/\langle\langle |V|\rangle\rangle_{\text{mw}} = L(\pi M/2kT)^{1/2}$; (ii) the average time interval separating consecutive pawl-peg bounces, i.e., $\langle\langle \Delta t\rangle\rangle_{\text{bounce}} = \Delta X_{\text{bounce}}/\langle\langle |V|\rangle\rangle_{\text{mw}} = L/[(1-e^{-A})\langle\langle |V|\rangle\rangle_{\text{mw}}] = \langle\langle \Delta t\rangle\rangle/(1-e^{-A}) = L(\pi M/2kT)^{1/2}/(1-e^{-A})$; (iii) $\Delta t'$, the time typically required for V to be changed appreciably compared with $\langle\langle |V|\rangle\rangle_{\text{mw}} = (2kT/\pi M)^{1/2}$ via DP-EBR X-directional *momentum* exchanges; and (iv) $\Delta t''$, the time, beginning immediately following a pawl-peg bounce and consequent reversal of $\operatorname{sgn} V$, typically required for the disk to (re)thermalize {i.e., for reversal of $\operatorname{sgn}[T_+(V)-T_-(V)]$} via DP-EBR X-directional *thermal-energy* exchanges.

Since $\Delta t'$ and $\Delta t''$ are, as per Appendix B of Ref. [3k] with improvements as per Appendixes B, F and I of this present paper — to first order in V/c — *in*dependent of V, they are not enclosed within single angular brackets. This *in*dependence implies that the time $\Delta t'$ [$\Delta t''$] required for $|V|$ [$|T_+(V)-T_-(V)| = 4T|V|/3c$] to decay to, say, $1/e$ of a given value is likewise *in*dependent of said given value, and is $\approx$ the time required for $|V|$ [$|T_+(V)-T_-(V)| = 4T|V|/3c$] to build up to $\langle\langle |V|\rangle\rangle_{\text{mw}} = (2kT/\pi M)^{1/2}$ [$\langle\langle |T_+(V)-T_-(V)|\rangle\rangle_{\text{mw}} = 4T\langle\langle |V|\rangle\rangle_{\text{mw}}/3c = 4T(2kT/\pi M)^{1/2}/3c$] via fluctuation (as per the fluctuation-dissipation theorem) [33].

We require: (a) $\Delta t''$ to be sufficiently short that $T_+(V)$ is the temperature of the $+X$ disk face (including the pawl) *itself* [12] — *not* merely of Doppler-shifted EBR "seen" thereby [13] — when V has a given value; and (b) that V change at most *very* slightly compared with $\langle\langle |V|\rangle\rangle_{\text{mw}} = (2kT/\pi M)^{1/2}$ via DP-EBR X-directional momentum exchanges *between* pawl-peg bounces [12]. Requirements (a) and (b) immediately above imply the first and last inequalities, respectively, in $\Delta t'' \lll \langle\langle \Delta t\rangle\rangle < \langle\langle \Delta t\rangle\rangle_{\text{bounce}} = \langle\langle \Delta t\rangle\rangle/(1-e^{-A}) \lll \Delta t'$. In the limit $F+R-1 \longrightarrow 1 \implies A \longrightarrow 0$, and to sufficient accuracy if $0 < A \ll 1$, $\langle\langle \Delta t\rangle\rangle_{\text{bounce}} = \langle\langle \Delta t\rangle\rangle/A$. If $A \longrightarrow 0 \implies \langle\langle \Delta t\rangle\rangle_{\text{bounce}} = \langle\langle \Delta t\rangle\rangle/A \longrightarrow \infty$, then the inequality in $\langle\langle \Delta t\rangle\rangle_{\text{bounce}} = \langle\langle \Delta t\rangle\rangle/A \lll \Delta t'$, and hence also Requirement (b) immediately above, is violated. Hence, we cannot let A become infinitesimal. But we can let A become *extremely* small before this becomes a problem. For, as per Appendix B of Ref. [3k] with improvements as per Appendixes B, F and I of this present paper, $\Delta t'/\Delta t'' \approx$ (DP rest-mass energy)/(DP thermal energy) $\approx Mc^2/MC^*T = Mc^2/CT = c^2/C^*T$, where C^* is the DP's specific heat per unit mass and $C = MC^*$ is its total heat capacity. For $0 < A \ll 1$, the inequality $\Delta t'' \ll \langle\langle\Delta t\rangle\rangle < \langle\langle\Delta t\rangle\rangle_{\text{bounce}} = \langle\langle\Delta t\rangle\rangle/(1-e^{-A}) \ll \Delta t'$ can be rewritten as $C^*T/c^2 \ll \langle\langle\Delta t\rangle\rangle/\Delta t' \ll \langle\langle\Delta t\rangle\rangle_{\text{bounce}}/\Delta t' = \langle\langle\Delta t\rangle\rangle/A\Delta t' \ll 1 \iff 1 \ll \langle\langle\Delta t\rangle\rangle/\Delta t'' \ll \langle\langle\Delta t\rangle\rangle_{\text{bounce}}/\Delta t'' = \langle\langle\Delta t\rangle\rangle/A\Delta t'' \ll c^2/C^*T$. In the limit $F+R-1 \longrightarrow 1 \implies A \longrightarrow 0$, the inequalities in $\langle\langle\Delta t\rangle\rangle_{\text{bounce}}/\Delta t' = \langle\langle\Delta t\rangle\rangle/A\Delta t' \ll 1 \iff \langle\langle\Delta t\rangle\rangle_{\text{bounce}}/\Delta t'' = \langle\langle\Delta t\rangle\rangle/A\Delta t'' \ll c^2/C^*T$, and hence also Requirement (b) immediately above, are violated, but not before A becomes *extremely* small — *much* smaller than A_{opt} corresponding to our *primary* objective of maximum power output and hence maximum time rate of negentropy production.

As the final items of *this* Appendix B, we (I) more quantitatively derive the requirement for negligibility of EBR drag acting on the DP moving through a distance L [which is essentially Requirement (b) immediately above if A is not too small compared with unity], and (II) more quantitatively derive Requirement (a) immediately above [previously (albeit less thoroughly) derived in Appendix B or Ref. [3k]] that the DP responds thermally in time $\Delta t'' \ll \langle\langle\Delta t\rangle\rangle$. Then, (III) we combine the results of (I) and (II) to recover (hopefully, somewhat more thoroughly), the requirement of Appendix B or Ref. [3k].

(I) Applying (2), and letting σ be the Stefan-Boltzmann constant and $a =$ (cross-sectional area of disk) $= [\pi\times$ (radius of disk)$^2 -$ (cross-sectional area of guide)]: Corresponding to V, the radiant power $r_\pm$ and radiant force $\mathcal{F}_\pm$ impinging on the $\pm X$ disk face — and thence $r_+ - r_-$ and $\mathcal{F}_+ - \mathcal{F}_-$, respectively — are [34]

$$
\begin{aligned}
& r_\pm = \sigma a T\left(V\right)_\pm^4 = \sigma a\left[T\left(1 \pm \frac{2V}{3c}\right)\right]^4 = \sigma a T^4\left(1 \pm \frac{8V}{3c}\right) \\
\implies\ & r_+ - r_- = \frac{16V}{3c}\sigma a T^4 \\
\implies\ & \mathcal{F}_\pm = -\frac{4}{3}\left(1 \pm \frac{V}{3c}\right)\frac{r_\pm}{c} = -\frac{4}{3}\frac{\sigma a T^4}{c}\left(1 \pm \frac{3V}{c}\right) \\
\implies\ & \mathcal{F}_+ - \mathcal{F}_- \equiv \mathcal{F}_{\text{net}} = -\frac{3}{2}\frac{r_+ - r_-}{c} = -\frac{8V}{c^2}\sigma a T^4 \\
\implies\ & \langle\langle|\mathcal{F}_{\text{net}}|\rangle\rangle_{\text{mw}} = \frac{8\langle\langle|V|\rangle\rangle_{\text{mw}}}{c^2}\sigma a T^4 = \frac{8}{c^2}\left(\frac{2kT}{\pi M}\right)^{1/2}\sigma a T^4.
\end{aligned}
\tag{B1}
$$

The lack of exact proportionality between the expressions for $r_\pm$ and $\mathcal{F}_\pm$ in (B1), i.e., the term $1 + \frac{V}{3c}$ in the expression for $\mathcal{F}_\pm$, arises because $\mathcal{F}_\pm$ entails an *extra* factor of $\cos\alpha$ [compare (B2) with (2)]:

$$
\begin{aligned}
\mathcal{F}_\pm &= \mathcal{F}_\pm(V) = \langle\mathcal{F}_\pm(V,\alpha)\rangle \\
&= \frac{\int_0^{\pi/2}\mathcal{F}_\pm(V,\alpha)\sin\alpha\cos\alpha d\alpha}{\int_0^{\pi/2}\sin\alpha\cos\alpha d\alpha} \\
&= \frac{\int_0^{\pi/2}\left[\frac{\sigma A T^4}{c}\left(1 \pm \frac{4V\cos\alpha}{c}\right)\cos\alpha\right]\sin\alpha\cos\alpha d\alpha}{\int_0^{\pi/2}\sin\alpha\cos\alpha d\alpha} \\
&= -\frac{4}{3}\frac{\sigma a T^4}{c}\left(1 \pm \frac{3V}{c}\right).
\end{aligned}
\tag{B2}
$$

But $\mathcal{F}_+ - \mathcal{F}_- \propto r_+ - r_-$, because this nonproportionality, i.e., the 1 in $1 + \frac{V}{3c}$, "subtracts out" when differences are taken.

Negligibility of EBR drag acting on the DP moving through a distance L requires that

$$\begin{aligned}
&\langle\langle \Delta t \rangle\rangle = \frac{L}{\langle\langle |V| \rangle\rangle_{\mathrm{mw}}} \ll \Delta t' \approx \frac{M \langle\langle |V| \rangle\rangle_{\mathrm{mw}}}{\langle\langle |\mathcal{F}_{\mathrm{net}}| \rangle\rangle_{\mathrm{mw}}} \\
&\Longrightarrow \frac{L}{\langle\langle |V| \rangle\rangle_{\mathrm{mw}}} \ll \frac{M \langle\langle |V| \rangle\rangle_{\mathrm{mw}}}{\frac{8\langle\langle |V| \rangle\rangle_{\mathrm{mw}}}{c^2}\sigma a T^4} \\
&\Longrightarrow \frac{L}{\langle\langle |V| \rangle\rangle_{\mathrm{mw}}} \ll \frac{Mc^2}{8\sigma a T^4} \\
&\Longrightarrow L \ll \frac{Mc^2}{8\sigma a T^4} \langle\langle |V| \rangle\rangle_{\mathrm{mw}} = \frac{Mc^2}{8\sigma a T^4} \left(\frac{2kT}{\pi M}\right)^{1/2} = \frac{M^{1/2}c^2}{8\sigma a T^{7/2}} \left(\frac{2k}{\pi}\right)^{1/2} \\
&\Longrightarrow \frac{LaT^{7/2}}{M^{1/2}} \ll \frac{c^2}{8\sigma} \left(\frac{2k}{\pi}\right)^{1/2} \iff \frac{\langle\langle \Delta t \rangle\rangle}{\Delta t'} = \frac{8\sigma a L}{c^2} \left(\frac{\pi T^7}{2kM}\right)^{1/2} \ll 1.
\end{aligned} \tag{B3}$$

Requirement (B3) [which is essentially Requirement (b) immediately above if A is not too small compared with unity] is easily fulfilled. [Of course, since $\mathcal{F}_{\mathrm{net}}$ fluctuates, it can sometimes (as per the fluctuation-dissipation theorem) cause $|V|$ to increase rather than decrease [33], but, given (B3), *all* changes in V and in $|V|$ caused by $\mathcal{F}_{\mathrm{net}}$ are negligible compared with those at pawl-peg bounces.]

(II) At an average pawl-peg collision, either disk *face* experiences, as per the paragraph containing (2) and that immediately following (24), a sudden temperature change in the EBR that it "sees" of

$$\langle\langle |\Delta T_{\mathrm{EBR}}| \rangle\rangle_{\mathrm{mw}} = \frac{4T \langle\langle |V| \rangle\rangle_{\mathrm{mw}}}{3c}. \tag{B4}$$

Thus, there must occur, in either disk *half* (of mass $\frac{M}{2}$), a temperature change of magnitude

$$\langle\langle |\Delta T_{\mathrm{DH}}| \rangle\rangle_{\mathrm{mw}} = \frac{1}{2} \langle\langle |\Delta T_{\mathrm{EBR}}| \rangle\rangle_{\mathrm{mw}} = \frac{2T \langle\langle |V| \rangle\rangle_{\mathrm{mw}}}{3c} \tag{B5}$$

in time $\Delta t'' \ll \langle\langle \Delta t \rangle\rangle = \frac{L}{\langle\langle |V| \rangle\rangle_{\mathrm{mw}}}$ immediately following any pawl-peg collision, if the DP is to respond thermally sufficiently rapidly so that $T_\pm(V)$ are essentially always the respective temperatures of the $\pm X$ disk faces themselves, and not merely that of the EBR "seen" thereby. Letting C^* be the DP's specific heat per unit mass (not to be confused with its total heat capacity $C = MC^*$), this requires that, in time $\Delta t'' \ll \langle\langle \Delta t \rangle\rangle = \frac{L}{\langle\langle |V| \rangle\rangle_{\mathrm{mw}}}$, energy of

$$|\Delta \mathfrak{E}_{\mathrm{DH}}| = \frac{M}{2} C^* \langle\langle |\Delta T_{\mathrm{DH}}| \rangle\rangle_{\mathrm{mw}} = \frac{MC^* T \langle\langle |V| \rangle\rangle_{\mathrm{mw}}}{3c} \tag{B6}$$

must be absorbed from the EBR by that ($+X$ or $-X$) disk *half* for which $\Delta T_{\mathrm{DH}} > 0$ at a given pawl-peg collision, and yielded to the EBR from the other disk *half*, for which $\Delta T_{\mathrm{DH}} > 0$ at a said pawl-peg collision.

During any pawl-peg collision, V switches sign. On average, $|V| = \langle\langle |V| \rangle\rangle_{\mathrm{mw}}$ both immediately before and immediately after a pawl-peg collision but with reversal of $\operatorname{sgn} V$, so that $\langle\langle |\Delta V| \rangle\rangle = 2 \langle\langle |V| \rangle\rangle_{\mathrm{mw}}$. Thus, the change in $r_+ - r_-$ resulting from a pawl-peg collision is of average magnitude

$$\langle\langle |\Delta (r_+ - r_-)| \rangle\rangle = \frac{16 \left(2 \langle\langle |V| \rangle\rangle_{\mathrm{mw}}\right)}{3c} \sigma a T^4 = \frac{32 \langle\langle |V| \rangle\rangle_{\mathrm{mw}}}{3c} \sigma a T^4. \tag{B7}$$

Hence, more quantitatively, our Requirement (a) of *this* Appendix B is:

$$\begin{aligned}
\Delta t'' &\approx \frac{|\Delta \mathfrak{E}_{\mathrm{DH}}|}{\langle\langle |r_+ - r_-|\rangle\rangle} \approx \frac{MC^*}{32\sigma a T^3} \ll \langle\langle \Delta t\rangle\rangle = \frac{L}{\langle\langle |V|\rangle\rangle_{\mathrm{mw}}} = L\left(\frac{\pi M}{2kT}\right)^{1/2} \\
&\Longrightarrow \frac{M^{1/2}C^*}{32\sigma a T^{5/2}} \ll L\left(\frac{\pi}{2k}\right)^{1/2} \\
&\Longrightarrow \frac{LaT^{5/2}}{M^{1/2}} \gg \frac{C^*}{32\sigma}\left(\frac{2k}{\pi}\right)^{1/2} \quad \Longleftrightarrow \quad \frac{\Delta t''}{\langle\langle \Delta t\rangle\rangle} = \frac{C^*}{32\sigma a L}\left(\frac{2kM}{\pi T^5}\right)^{1/2} \ll 1.
\end{aligned} \tag{B8}$$

(III) Combining Requirements (B3) and (B8) = (a) of *this* Appendix B yields the easily-fulfilled requirement previously (albeit somewhat less thoroughly) derived in Appendix B of Ref. [3k]:

$$\frac{\Delta t''}{\Delta t'} = \frac{\Delta t''}{\langle\langle \Delta t\rangle\rangle}\frac{\langle\langle \Delta t\rangle\rangle}{\Delta t'} = \left[\frac{C^*}{32\sigma a L}\left(\frac{2kM}{\pi T^5}\right)^{1/2}\right]\left[\frac{8\sigma a L}{c^2}\left(\frac{\pi T^7}{2kM}\right)^{1/2}\right] = \frac{C^* T}{4c^2} \ll 1. \tag{B9}$$

Appendix C: (Re)thermalization and Z-directional Brownian motion of the pawl

Fluctuations of the pawl's altitude Z obtain *mainly* via its intermolecular Z-directional momentum and kinetic-energy exchanges with the $+X$ disk face, which is a "*local* heat bath" at temperature $T_+(V)$ for the pawl when the DP's X-directional Brownian-motional velocity happens to be V; by comparison, pawl-EBR Z-directional momentum and kinetic-energy exchanges are negligible. For simplicity, let pawl/$+X$-disk-face Z-directional momentum and kinetic-energy exchanges — and hence the pawl's sampling of its Boltzmann distribution corresponding to $T_+(V)$ as per (3) – (6) — occur *mainly* at pawl-stop bounces when $Z = Z_{\min}$: the stop is *within the* $+X$ *disk face* and hence part of said "*local* heat bath" at temperature $T_+(V)$ [12,13]. [Of course, Doppler-shifted EBR at temperature $T_+(V)$ [13] is the *primary* element of said "*local* heat bath", with said temperature $T_+(V)$ then obtaining for the $+X$ disk face (including the stop and the pawl) *itself* [12] via $+X$-disk-face/EBR X*-directional thermal-energy* exchanges when the DP's X-directional Brownian-motional velocity happens to be V. But pawl-EBR Z*-directional momentum and kinetic-energy* exchanges are negligible compared with pawl/$+X$-disk-face — specifically, pawl-stop — Z*-directional momentum and kinetic-energy* exchanges.] $A \equiv mgH_{\mathrm{net}}/kT$ can easily be (i) large enough so that $\langle\langle \Delta t\rangle\rangle_{\mathrm{bounce}} = \langle\langle \Delta t\rangle\rangle/(1-e^{-A}) \ll \Delta t' \implies \langle\langle \Delta t\rangle\rangle \ll (1-e^{-A})\Delta t' \approx \Delta t'$ (recall the fifth paragraph of Sect. 4, and Appendix B), yet (ii) small enough so that $H_{\mathrm{net}} \ll L$ [12]. (i) immediately above $\implies e^A \gg 1/(1-\langle\langle \Delta t\rangle\rangle/\Delta t') \implies A > -\ln(1-\langle\langle \Delta t\rangle\rangle/\Delta t') \overset{\langle\langle \Delta t\rangle\rangle \ll \Delta t'}{\Longrightarrow} A > \langle\langle \Delta t\rangle\rangle/\Delta t'$, which can easily be consistent with (ii) immediately above. Let v be the pawl's (non-relativistic) Z-directional Brownian-motional velocity. Averaging over v's one-dimensional Maxwellian probability density $P(v)_{\mathrm{mw}} = (m/2\pi kT)^{1/2}\exp(-mv^2/2kT)$ yields $\langle |v|\rangle_{\mathrm{mw}} = (2kT/\pi m)^{1/2}$ — which, of course, exceeds $\langle\langle |V|\rangle\rangle_{\mathrm{mw}} = (2kT/\pi M)^{1/2}$ by the ratio $(M/m)^{1/2}$. [Enclosure within single angular brackets denoting averaging over *all* v should *not* be confused with denotation of averages over any *one* given $\pm|V|$ pair (over *all* $|V|$) via enclosure within single (double) angular brackets as per the third paragraph and the last four paragraphs of Sect. 2. Also, negligible error results from employing $P(v)_{\mathrm{mw}} = (m/2\pi kT)^{1/2}\exp(-mv^2/2kT)$, rather than $P(v|V)_{\mathrm{mw}} = [m/2\pi kT_+(V)]^{1/2}\exp[-mv^2/2kT_+(V)]$.] Let

$\Delta t''' = H_{\rm net}/\left|v\right|$: its average is $\langle \Delta t''' \rangle = H_{\rm net}/\left\langle |v| \right\rangle_{\rm mw} = H_{\rm net}(\pi m/2kT)^{1/2}$. Hence, $\langle \Delta t''' \rangle / \langle\langle \Delta t \rangle\rangle = (H_{\rm net}/L)(m/M)^{1/2}$, which is $\ll 1$ given $H_{\rm net} \ll L$ and $m \ll M$ — so that the pawl has ample time to (re)thermalize between consecutive pawl-peg interactions.

The pawl is thus a one-Brownian-particle "isothermal atmosphere" in *local TEQ* [12,13] at temperature $T_+(V)$ [12,13] when the DP's X-directional Brownian-motional velocity happens to be V.

Appendix D: Justification of the second step of (18)

Restating the first two lines of (18),

$$\begin{aligned} \langle P^* \rangle_{N+\frac{1}{2}} &= \langle fV \rangle_{N+\frac{1}{2}} \\ &= \langle f \rangle_{N+\frac{1}{2}} \langle V \rangle_{N+\frac{1}{2}} \, . \end{aligned} \tag{D1}$$

We wish to justify the second steps of (18) and (D1).

The most general circumstance under which $\langle xy \rangle = \langle x \rangle \langle y \rangle$ is the vanishing of the covariance [35]

$$\begin{aligned} \mathrm{Cov}(x,y) &\equiv \langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle \\ &= \langle xy \rangle - \langle x \rangle \langle y \rangle \, . \end{aligned} \tag{D2}$$

(If the second moments of both x and y are finite, i.e., if $0 < \mathrm{Var}(x) \equiv (x - \langle x \rangle)^2 = \langle x^2 \rangle - \langle x \rangle^2 < \infty$ and $0 < \mathrm{Var}(y) \equiv (y - \langle y \rangle)^2 = \langle y^2 \rangle - \langle y \rangle^2 < \infty$, then the correlation coefficient — which has the advantage of being dimensionless even if x and y are not — can be substituted for the covariance [35].)

Letting $\langle y \rangle |x_i$ be the *conditional* expectation value of y given that $x = x_i$, we have

$$\begin{aligned} & x \text{ and } y \text{ are } \textit{completely in}\text{dependent} \\ \Longrightarrow \;& \langle y \rangle |x_i \text{ is } \textit{in}\text{dependent of } x_i \\ \Longrightarrow \;& \mathrm{Cov}(x,y) = 0 \iff \langle xy \rangle = \langle x \rangle \langle y \rangle \, . \end{aligned} \tag{D3}$$

The first, second, and third lines of (D3) are successively less restrictive special cases. In contrast with the first line of (D3), the second line of (D3) allows *some* dependence of y on x, i.e., the dependence on x_i of any moment $\langle y^n \rangle |x_i$ except the first ($n = 1$).

In accordance with (D3) — especially the second line of (D3) — we have, for any two discrete random variables x and y for which $\langle y \rangle |x_i$ is *in*dependent of x_i,

$$\begin{aligned} \langle xy \rangle &= \sum_i \sum_j x_i y_j P(x_i y_j) \\ &= \sum_i \sum_j x_i y_j P(x_i) P(y_j|x_i) \\ &= \sum_i \left[x_i P(x_i) \sum_j y_j P(y_j|x_i) \right] \\ &= \sum_i x_i P(x_i) \langle y \rangle |x_i \\ &= \langle y \rangle \sum_i x_i P(x_i) \\ &= \langle x \rangle \langle y \rangle \, . \end{aligned} \tag{D4}$$

The fifth step of (D4) is justified given that $\langle y\rangle = \langle y\rangle |x_i$ is *in*dependent of x_i. (The result for continuous random variables x and y, as well as for those possessing both discrete and continuous ranges, is, of course, similar.)

Now, let $x = f_{N+\frac{1}{2}} = \Delta p_{N+\frac{1}{2}} \big/ \Delta t$ and $y = V_{N+\frac{1}{2}}$ ($= +|V|$ for a $+ \longrightarrow +$ pawl-over-peg jump, $-|V|$ for a $- \longrightarrow -$ pawl-over-peg jump, and 0 for $- \longrightarrow +$ and $+ \longrightarrow -$ pawl-peg bounces). The second steps of (18) and (D1) are justified, in light of the immediately preceding paragraph [especially (D4)], because $\langle V\rangle_{N+\frac{1}{2}}$ of (15) — as opposed to *individual* values of $V_{N+\frac{1}{2}}$ as per the immediately preceding sentence — is *in*dependent of which of the four *individual* values of $\Delta p_{N+\frac{1}{2}}$ ($\Delta p_{+\longrightarrow+} = 0$, $\Delta p_{-\longrightarrow-} = 0$, $\Delta p_{-\longrightarrow+} = 2M|V|$, or $\Delta p_{+\longrightarrow-} = -2M|V|$), and hence of the corresponding $f_{N+\frac{1}{2}} = \Delta p_{N+\frac{1}{2}} \big/ \Delta t$, that happens to occur at the $N \longrightarrow N+1$ transition, i.e., at the $(N+1)\,\mathrm{st}$ pawl-peg interaction.

Finally, since the second steps of (18) and (D1) are justified for *all* $\pm|V|$ pairs, averaging over *all* $\pm|V|$ pairs, i.e., over *all* $|V|$, yields

$$\begin{aligned}\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}} &= \langle\langle fV\rangle\rangle_{N+\frac{1}{2}} \\ &= \langle\langle f\rangle\rangle_{N+\frac{1}{2}} \langle\langle V\rangle\rangle_{N+\frac{1}{2}}\,. \end{aligned} \tag{D5}$$

Also, see our alternative derivation of $\langle f\rangle_{N+\frac{1}{2}}$ and $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$, and our derivations of related quantities, in Appendix E.

Appendix E: Alternative derivation of $\langle f\rangle_{N+\frac{1}{2}}$ and $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$, and excess pawl-peg bounce probability

Only pawl-peg bounces, and not pawl-over-peg jumps, exert force on the DP. Considering first any *one* given $\pm|V|$ pair, the DP experiences a $- \longrightarrow +$ [$+ \longrightarrow -$] pawl-peg bounce and corresponding X-directional momentum change of $\Delta p_{-\longrightarrow+} = 2M|V|$ [$\Delta p_{+\longrightarrow-} = -2M|V|$], with probability $P(<-)_N$ [$P(<+)_N$], at the $N \longrightarrow N+1$ transition, i.e., at the $(N+1)\,\mathrm{st}$ pawl-peg interaction. Consecutive pawl-peg interactions are separated in time by $\Delta t = L/|V|$. (Recall Appendix A.) Thus, by Newton's second law, we have the following alternative derivation of $\langle f\rangle_{N+\frac{1}{2}}$ and $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$:

$$\begin{aligned}
\langle f\rangle_{N+\frac{1}{2}} &= \langle \Delta p\rangle_{N+\frac{1}{2}} \Big/ \Delta t \\
&= [\Delta p_{-\longrightarrow+}P(<-)_N + \Delta p_{+\longrightarrow-}P(<+)_N]/\Delta t \\
&= [\Delta p_{-\longrightarrow+}P(-)_N P(<|-) + \Delta p_{+\longrightarrow-}P(+)_N P(<|+)]/\Delta t \\
&= 2M\,|V|\,[P(-)_N P(<|-) - P(+)_N P(<|+)]/\Delta t \\
&= 2M\,|V|\,[P(-)_N(1-R) - P(+)_N(1-F)]/\Delta t \\
&= 2M\,|V|\,\{[1-P(+)_N](1-R) - P(+)_N(1-F)\}/\Delta t \\
&= 2M\,|V|\,[1-R-(2-F-R)P(+)_N]/\Delta t \\
&= M\,|V|\,(F-R)(F+R-1)^N/\Delta t \\
&= M\,|V|\,(F-R)(F+R-1)^N/(L/\,|V|) \\
&= (MV^2/L)(F-R)(F+R-1)^N \\
&= (4M\,|V|^3/3Lc)Ae^{-A}(2e^{-A}-1)^N \\
\Longrightarrow\ \langle\langle f\rangle\rangle_{N+\frac{1}{2}} &= (4M\,\langle\langle|V|^3\rangle\rangle_{\text{mw}} \big/\, 3Lc)Ae^{-A}(2e^{-A}-1)^N \\
&= 4\{[2(kT)^3/M]^{1/2}/3Lc\}Ae^{-A}(2e^{-A}-1)^N. \qquad \text{(E1)}
\end{aligned}$$

In the fifth step of (E1), we applied (9c) and (9d); in the sixth step, $P(+)_N + P(-)_N = 1$; in the eighth step, the eighth line of (10); and in the ninth step, $\Delta t = L/\,|V|$. The tenth line of (E1) agrees with (17) for *general* F and R that [as per the sentence immediately following (9d)] are *at most* functions of $|V|$ *only* — and hence *constant* for any *one* given $|V|$; and the eleventh line of (E1) agrees with (21) for the *specific* F and R given by the rightmost terms of (5) and (6), respectively. The last two lines of (E1) are justified by the paragraph immediately following (24), and yield $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$, in agreement with (27).

Given $V = \pm\,|V|$, the excess probability ξ_N of $- \longrightarrow +$ pawl-peg bounces over $+ \longrightarrow -$ pawl-peg bounces is

$$\begin{aligned}
\xi_N &= P(<-)_N - P(<+)_N \\
&= P(-)_N P(<|-) - P(+)_N P(<|+) \\
&= P(-)_N(1-R) - P(+)_N(1-F) \\
&= [1-P(+)_N](1-R) - P(+)_N(1-F) \\
&= 1-R-(2-F-R)P(+)_N \\
&= \frac{1}{2}(F-R)(F+R-1)^N \\
&= (2\,|V|\,/3c)Ae^{-A}(2e^{-A}-1)^N \\
\Longrightarrow\ \langle\langle \xi\rangle\rangle_N &= (2\,\langle\langle|V|\rangle\rangle_{\text{mw}}\,/3c)Ae^{-A}(2e^{-A}-1)^N \\
&= [2(2kT/\pi M)^{1/2}/3c]Ae^{-A}(2e^{-A}-1)^N. \qquad \text{(E2)}
\end{aligned}$$

In the third step of (E2), we applied (9c) and (9d); in the fourth step, $P(+)_N + P(-)_N = 1$; and in the sixth step, the eighth line of (10). The sixth line of (E2) is valid for *general* F and R that [as per the sentence immediately following (9d)] are *at most* functions of $|V|$ *only* — and hence *constant* for any *one* given $|V|$;

and the seventh line of (E2) is valid for the *specific* F and R given by the rightmost terms of (5) and (6), respectively. [Since, considering any *one* given $\pm|V|$ pair, ξ_N as expressed in (E2) does not involve an average, ξ_N is not enclosed within single angular brackets as $\langle\xi\rangle_N$.] The last two lines of (E2) are justified by the paragraph immediately following (24). ξ_N and $\langle\langle\xi\rangle\rangle_N$ are maximized by: (a) first, setting $\xi_N = \xi_0$ ($\langle\langle\xi\rangle\rangle_N = \langle\langle\xi\rangle\rangle_0$), thereby maximizing $(F+R-1)^N$ at unity $\Longrightarrow$ maximizing $(2e^{-A}-1)^N$ at unity; and (b) then, setting $d(Ae^{-A})/dA = 0 \Longrightarrow A = 1 \Longrightarrow (Ae^{-A})_{\max} = e^{-1}$. We thus obtain the *absolute* maxima $\xi_{N,\max} = \xi_0|(A=1) = 2|V|/3ec \Longrightarrow \langle\langle\xi\rangle\rangle_{N,\max} = \langle\langle\xi\rangle\rangle_0|(A=1) = 2\langle\langle|V|\rangle\rangle_{\text{mw}}/3ec = 2(2kT/\pi M)^{1/2}/3ec$. By contrast, $\xi_\infty = 0 \Longrightarrow \langle\langle\xi\rangle\rangle_\infty = 0$. (If $F+R-1=0 \Longrightarrow A = \ln 2$, then ξ_∞ and $\langle\langle\xi\rangle\rangle_\infty$ are reached at $N=1$, i.e., then $\xi_1 = \xi_\infty = 0 \Longrightarrow \langle\langle\xi\rangle\rangle_1 = \langle\langle\xi\rangle\rangle_\infty = 0$.)

$\left(\langle f\rangle_{N+\frac{1}{2}} + \langle f\rangle_{N+\frac{3}{2}}\right)\Big/2$ for all even $N \geq 0$ if $F+R-1 \neq 0 \Longrightarrow A \neq \ln 2$, and $\langle f\rangle_{N+\frac{1}{2}}$ for all $N \geq 0$ if $0 < F+R-1 < 1 \Longrightarrow \ln 2 > A > 0$, decreases monotonically and asymptotically towards 0 with increasing N, because the excess probability ξ_N of $- \longrightarrow +$ pawl-peg bounces over $+ \longrightarrow -$ pawl-peg bounces does likewise, finally vanishing as $N \longrightarrow \infty$ (at $N=1$ if $F+R-1=0 \Longrightarrow A = \ln 2$). (Of course, the behavior of $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$ with respect to $\langle\langle\xi\rangle\rangle_N$ is similar.) Once $\xi_N = 0 \Longrightarrow \langle\langle\xi\rangle\rangle_N = 0$, $- \longrightarrow +$ pawl-peg bounces are no longer any more probable than $+ \longrightarrow -$ pawl-peg bounces, hence implying $\langle f\rangle_{N+\frac{1}{2}} = 0 \Longleftrightarrow d\langle V\rangle_N/dt = 0 \Longrightarrow \langle\langle f\rangle\rangle_{N+\frac{1}{2}} = 0 \Longleftrightarrow d\langle\langle V\rangle\rangle_N/dt = 0$ — and thus that $\langle V\rangle_N$ and $\langle\langle V\rangle\rangle_N$ can increase no further.

Comparison of the first and eighth lines of (E1), and applying first (5) and (6) and then the paragraph immediately following (24), yields

$$\begin{aligned}
\langle\Delta p\rangle_{N+\frac{1}{2}} &= M|V|(F-R)(F+R-1)^N \\
&= (4MV^2/3c)Ae^{-A}(2e^{-A}-1)^N \\
\Longrightarrow \langle\langle\Delta p\rangle\rangle_{N+\frac{1}{2}} &= \left(4M\left\langle\left\langle V^2\right\rangle\right\rangle_{\text{mw}}\Big/3c\right)Ae^{-A}(2e^{-A}-1)^N \\
&= (4kT/3c)Ae^{-A}(2e^{-A}-1)^N. \qquad \text{(E3)}
\end{aligned}$$

Maximization of $\langle\Delta p\rangle_{N+\frac{1}{2}}$ ($\langle\langle\Delta p\rangle\rangle_{N+\frac{1}{2}}$) is identical with that of $\langle f\rangle_{N+\frac{1}{2}}$ ($\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$) [as per the paragraph containing (32)] and of ξ_N ($\langle\langle\xi\rangle\rangle_N$) [as per the discussion following (E2)], yielding the *absolute* maxima $\langle\Delta p\rangle_{N+\frac{1}{2},\max} = \langle\Delta p\rangle_{\frac{1}{2}}\Big|(A=1) = 4MV^2/3ec \Longrightarrow \langle\langle\Delta p\rangle\rangle_{N+\frac{1}{2},\max} = \langle\langle\Delta p\rangle\rangle_{\frac{1}{2}}\Big|(A=1) = 4M\langle\langle V^2\rangle\rangle_{\text{mw}}/3ec = 4kT/3ec$.

Comparing the sixth line of (E2) with the first line of (E3) [or the seventh line of (E2) with the second line of (E3)] yields

$$\xi_N = \langle\Delta p\rangle_{N+\frac{1}{2}}\Big/2M|V|. \qquad \text{(E4)}$$

[Since the fundamental expression for ξ_N as per (E2) does not involve an average, ξ_N is not enclosed within single angular brackets in (E4).] Comparing the respective second-to-the-last or last lines of (E2) and (E3) yields

$$\begin{aligned}
\langle\langle\xi\rangle\rangle_N &= \langle\langle\Delta p\rangle\rangle_{N+\frac{1}{2}}\langle\langle|V|\rangle\rangle_{\text{mw}}\Big/2M\left\langle\left\langle V^2\right\rangle\right\rangle_{\text{mw}} \\
&= \langle\langle\Delta p\rangle\rangle_{N+\frac{1}{2}}(2kT/\pi M)^{1/2}\Big/[2M(kT/M)] \\
&= \langle\langle\Delta p\rangle\rangle_{N+\frac{1}{2}}\Big/(2\pi MkT)^{1/2}. \qquad \text{(E5)}
\end{aligned}$$

Time evolution of ξ_N and $\langle\langle\xi\rangle\rangle_N$ and of $\langle\Delta p\rangle_{N+\frac{1}{2}}$ and $\langle\langle\Delta p\rangle\rangle_{N+\frac{1}{2}}$ — like that of $\langle f\rangle_{N+\frac{1}{2}}$ and $\langle\langle f\rangle\rangle_{N+\frac{1}{2}}$ — towards 0 as $N \longrightarrow \infty$ is monotonic and asymptotic through positive values for all $N \geq 0$ if $0 < F + R - 1 < 1 \implies \ln 2 > A > 0$, diminishing-oscillatory through positive (negative) values at all even (odd) $N \geq 0$ if $-1 < F + R - 1 < 0 \implies \infty > A > \ln 2$, and complete at $N = 1$ if $F + R - 1 = 0 \implies A = \ln 2$.

Appendix F: EBR radiant power and force, and Carnot-engine DP power outputs

In this Appendix F, conceptual, notational, and numerical corrections, as per Appendix I, are incorporated into a review of EBR radiant power, of EBR radiant force, and thence of the DP's Carnot-engine power outputs — in improvement of Appendixes B and C [17] of Ref. [3k]. [For simplicity, in this Appendix F, we assume the *specific* F and R given by the rightmost terms of (5) and (6), respectively, and, as usual, accuracy to first order in V/c.]

In this Appendix F, we first consider Case I: the DP's Carnot-engine power output given *slow* Carnot cycles, each requiring time $\gg \Delta t_{\text{bounce}}$ of Appendix B: In Case I, the DP Carnot engine "sees" $T_+(V) - T_-(V)$ to be "averaged" or "smoothed-out" over many reversals of $\operatorname{sgn} V$. Then we consider Case II: the DP's Carnot-engine power output given *fast* Carnot cycles, each requiring time $\ll \Delta t_{\text{bounce}}$ of Appendix B: In Case II, the DP Carnot engine "sees" $T_+(V) - T_-(V)$ to be reversed with each reversal of $\operatorname{sgn} V$. [Fast real-world (classical) Carnot cycles are typically less likely to approach the (classical) theoretical-maximum Carnot efficiency than slow ones [36], but for simplicity, we neglect this technical — not fundamental — difficulty in this Appendix F: (If necessary, we simply assume that even our "fast" cycles are slow enough).]

Case I was discussed (albeit semiquantitatively, without consideration of time evolution) in Appendix C [17] of Ref. [3k]. Case II is new to this present paper.

Appendix F, Case I:

We obtain, for the *average* temperature difference between the $+X$ and $-X$ disk faces,

$$
\begin{aligned}
&T_+(V) - T_-(V) = T\left(1 + \frac{2V}{3c}\right) - T\left(1 - \frac{2V}{3c}\right) = 4TV/3c \\
&\implies \langle T_+(V) - T_-(V)\rangle_N = T\left(1 + \frac{2\langle V\rangle_N}{3c}\right) - T\left(1 - \frac{2\langle V\rangle_N}{3c}\right) = 4T\langle V\rangle_N/3c \\
&= (8TV^2/9c^2)A[1 - (2e^{-A} - 1)^N]/(e^A - 1) \\
&\implies \langle\langle T_+(V) - T_-(V)\rangle\rangle_N = T\left(1 + \frac{2\langle\langle V\rangle\rangle_N}{3c}\right) - T\left(1 - \frac{2\langle\langle V\rangle\rangle_N}{3c}\right) \\
&= 4T\langle\langle V\rangle\rangle_N/3c \\
&= 8T(\langle\langle V^2\rangle\rangle_{\text{mw}}/9c^2)A[1 - (2e^{-A} - 1)^N]/(e^A - 1) \\
&= (8kT^2/9Mc^2)A[1 - (2e^{-A} - 1)^N]/(e^A - 1) \qquad \text{(F1)}
\end{aligned}
$$

— as per (2), the two sentences immediately following (2), and the paragraph immediately thereafter, and then application of (19) and (25), respectively, to (2) for the DP's *average* motion over any *one* given $\pm|V|$ pair and over *all* $|V|$, respectively.

In numerical correction of the first paragraph of Appendix B of Ref. [3k] [including Eqs. (B1) and (B2) thereof]: The radiant power $r_\pm$ and radiant force $\mathcal{F}_\pm$ impinging on the $\pm X$ disk face is, upon applying

(2) and (F1), and letting σ be the Stefan-Boltzmann constant and $a =$ (cross-sectional area of disk) $= [\pi\times$ (radius of disk)$^2 -$ (cross-sectional area of guide)],

$$
\begin{aligned}
r_{\pm} &= \sigma a T_{\pm}^4 = \sigma a \left[T\left(1 \pm \frac{2V}{3c}\right)\right]^4 = \sigma a T^4 \left(1 \pm \frac{8V}{3c}\right) \\
&\implies r_+ - r_- = \frac{16V}{3c}\sigma a T^4 \\
&\implies \langle r_+ - r_- \rangle_N = \frac{16\,\langle V\rangle_N}{3c}\sigma a T^4 = \frac{32V^2 A[1-(2e^{-A}-1)^N]}{9c^2(e^A-1)}\sigma a T^4 \\
&\implies \langle\langle r_+ - r_- \rangle\rangle_N = \frac{16\,\langle\langle V\rangle\rangle_N}{3c}\sigma a T^4 = \frac{32\,\langle\langle V^2\rangle\rangle_{\mathrm{mw}}\, A[1-(2e^{-A}-1)^N]}{9c^2(e^A-1)}\sigma a T^4 \\
&= \frac{32kTA[1-(2e^{-A}-1)^N]}{9Mc^2(e^A-1)}\sigma a T^4 \\
&\implies \mathcal{F}_{\pm} = -\frac{4}{3}\left(1 \pm \frac{V}{3c}\right)\frac{r_{\pm}}{c} = -\frac{4}{3}\frac{\sigma a T^4}{c}\left(1 \pm \frac{3V}{c}\right) \\
&\implies \mathcal{F}_+ - \mathcal{F}_- \equiv \mathcal{F}_{\mathrm{net}} = -\frac{3}{2}\frac{r_+ - r_-}{c} = -\frac{8V}{c^2}\sigma a T^4 \\
&\implies \langle \mathcal{F}_{\mathrm{net}} \rangle_N = -\frac{8\,\langle V\rangle_N}{c^2}\sigma a T^4 = -\frac{16V^2 A[1-(2e^{-A}-1)^N]}{3c^3(e^A-1)}\sigma a T^4 \\
&\implies \langle\langle \mathcal{F}_{\mathrm{net}} \rangle\rangle_N = -\frac{8\,\langle\langle V\rangle\rangle_N}{c^2}\sigma a T^4 = -\frac{16\,\langle\langle V^2\rangle\rangle_{\mathrm{mw}}\, A[1-(2e^{-A}-1)^N]}{3c^3(e^A-1)}\sigma a T^4 \\
&= -\frac{16kTA[1-(2e^{-A}-1)^N]}{3Mc^3(e^A-1)}\sigma a T^4 = -\frac{16\sigma a k T^5 A[1-(2e^{-A}-1)^N]}{3Mc^3(e^A-1)}.
\end{aligned}
\tag{F2}
$$

The minus sign obtains in $\mathcal{F}_{\mathrm{net}}$ because $\mathcal{F}_{\mathrm{net}}$ tends to decrease $|V|$ (and likewise for $\langle \mathcal{F}_{\mathrm{net}} \rangle_N$ and $\langle\langle \mathcal{F}_{\mathrm{net}} \rangle\rangle_N$). [The sixth and seventh lines of (F2) are justified as per the paragraph containing (B1) and (B2).] Since, corresponding to V, the DP's X-directional momentum is $p = MV$, $\mathcal{F}_{\mathrm{net}}$ tends to decrease $|V|$ in an e-folding time scale of

$$
\Delta t' = \left|\frac{p}{\mathcal{F}_{\mathrm{net}}}\right| = \frac{MV}{\frac{8V}{c^2}\sigma a T^4} = \frac{Mc^2}{8\sigma a T^4}, \tag{F3}
$$

which is *in*dependent of V. [Of course, because fluctuations in V at TEQ are time-symmetrical, $|V|$ will (as per the fluctuation-dissipation theorem) sometimes increase — but typically only in a like e-building time scale of $\Delta t'$ [33].]

Note that — as per (F1), (F2), and the associated discussions [and since, as per the paragraph containing Eq. (B3) of Ref. [3k], the (re)thermalization energy change of the $+X$ or $-X$ disk *half* is $\propto \langle\langle T_+(V) - T_-(V)\rangle\rangle$] — the numerator and denominator of Eq. (B3) of Ref. [3k] both require numerical correction by the *identical* factor of $2/3$ — hence, the rightmost term thereof is unchanged since these two numerical corrections therein cancel out.

Also note — accounting for differences in notation [13a,34] — that $\mathcal{F}_{\mathrm{net}}$ is $4/3$ times that derived by Peebles [13a,34]. This obtains because our $\pm X$ disk faces are *flat* $\pm X$-directional EBR receptors, whereas a *spherical* particle (as considered by Peebles [13a,34]) has *hemispherical* $\pm X$-directional EBR receptors.

Since EBR impinges perpendicularly ($\alpha = 0 \iff \cos\alpha = 1$) *onto a sphere from any and all directions*, for a sphere's *hemispherical* $\pm X$-directional EBR receptors, (2) is modified to

$$
\begin{aligned}
T_{\pm}(V) &= \langle T_{\pm}(V,\alpha)\rangle \\
&= \frac{\int_0^{\pi/2} T\left(1 \pm \frac{V\cos\alpha}{c}\right)\sin\alpha d\alpha}{\int_0^{\pi/2}\sin\alpha d\alpha} \\
&= T\left(1 \pm \frac{V}{2c}\right).
\end{aligned}
\tag{F4}
$$

Hence, $T_{+}(V) - T_{-}(V) = V/c$ for *hemispherical* $\pm X$-directional EBR receptors — as opposed to $T_{+}(V) - T_{-}(V) = 4V/3c$ for *flat* $\pm X$-directional EBR receptors. Since $\mathcal{F}_{\text{net}} \propto T_{+}(V) - T_{-}(V)$, said factor of $4/3$ is explained.

It should be pointed out that said factor of $4/3$ obtains *only* given that — as in our system — the DP is constrained to *one*-dimensional (X-directional) motion by the guide. If the DP were *not* so constrained, so that it could manifest rotational as well as translational Brownian motion, then the DP would present a spherical cross-section on the average, and hence said factor of $4/3$ would *not* obtain. [Of course, $\operatorname{sgn}(T_{+} - T_{-})$ would then *not* be correlated with the direction of DP motion, thus contravening our challenge to the second law.]

Applying (F1) and (F2), the DP's "secondary" (Carnot-efficiency) power output is

$$
\begin{aligned}
&\langle P^{*}_{\text{Carnot}}\rangle_N = \langle r_{+} - r_{-}\rangle_N \times \langle \text{Carnot efficiency}\rangle_N \\
&= \langle r_{+} - r_{-}\rangle_N \times \frac{\langle T_{+}(V) - T_{-}(V)\rangle_N}{\langle T_{+}(V)\rangle_N} \doteq \langle r_{+} - r_{-}\rangle_N \times \frac{\langle T_{+}(V) - T_{-}(V)\rangle_N}{T} \\
&= \frac{16\,\langle V\rangle_N}{3c}\sigma a T^4 \times \frac{4T\,\langle V\rangle_N/3c}{T} = \frac{64}{9}\sigma a T^4 \frac{(\langle V\rangle_N)^2}{c^2} \\
&= \frac{64}{9c^2}\sigma a T^4 \left\{\frac{2V^2 A[1 - (2e^{-A} - 1)^N]}{3c(e^A - 1)}\right\}^2 \\
&= \frac{256}{81}\sigma a T^4 \frac{V^4}{c^4}\left\{\frac{A[1 - (2e^{-A} - 1)^N]}{(e^A - 1)}\right\}^2 \\
&= \langle|\mathcal{F}_{\text{net}}|\rangle_N \langle V\rangle_N \\
&\implies \langle\langle P^{*}_{\text{Carnot}}\rangle\rangle_N = \frac{64}{9}\sigma a T^4 \frac{\langle\langle V\rangle^2\rangle_N}{c^2} \\
&= \frac{256}{81}\sigma a T^4 \frac{\langle\langle V^4\rangle\rangle_{\text{mw}}}{c^4}\left\{\frac{A[1 - (2e^{-A} - 1)^N]}{(e^A - 1)}\right\}^2 \\
&= \frac{256}{27}\sigma a T^4 \frac{(kT/M)^2}{c^4}\left\{\frac{A[1 - (2e^{-A} - 1)^N]}{(e^A - 1)}\right\}^2 \\
&= \frac{256}{27}\sigma a T^4 \frac{\langle\langle V^2\rangle\rangle^2_{\text{mw}}}{c^4}\left\{\frac{A[1 - (2e^{-A} - 1)^N]}{(e^A - 1)}\right\}^2 \\
&= \langle|\mathcal{F}_{\text{net}}|\rangle_N \langle\langle V\rangle\rangle_N .
\end{aligned}
\tag{F5}
$$

The second step in the second line of (F5) is justified because V is nonrelativistic, with $|V| \ll c$ for all values of $|V|$ that have nonnegligible probabilities of being equaled or exceeded. Since $\langle T_+(V) - T_-(V)\rangle_N \propto \langle V\rangle_N \implies \langle\langle T_+(V) - T_-(V)\rangle\rangle_N \propto \langle\langle V\rangle\rangle_N$, time evolution and maximization of $\langle T_+(V) - T_-(V)\rangle_N$ [$\langle\langle T_+(V) - T_-(V)\rangle\rangle_N$] is identical with that of $\langle V\rangle_N$ [$\langle\langle V\rangle\rangle_N$] [as per (2), (11) – (14) and the associated discussions, (19), (25), and the first five paragraphs of Sect. 4 — but recalling "practical" limits as per the fifth paragraph of Sect. 4 and as per Appendix B]. Time evolution and maximization of $\langle P^*_{\text{Carnot}}\rangle_N$ and $\langle\langle P^*_{\text{Carnot}}\rangle\rangle_N$ are somewhat faster than of $\langle P^*\rangle_N$ and $\langle\langle P^*\rangle\rangle_N$, respectively, owing to [compare (F5) with (22) and (28)]: (a) $\langle V\rangle_N$ — and hence, specifically, $[1 - (2e^{-A} - 1)^N]$ — being *squared* in (F5) and (b) $\left(2e^{-A} - 1\right)^N$ *not* being multiplied by the prefactor $e^{-A} < 1$ in (F5). [Otherwise, said two time evolutions are similar (identical if $F + R - 1 = 0 \implies A = \ln 2 \implies$ completion of time evolution at $N = 1$ — as obtains for *all* quantities studied in this present paper).]

Note, as per the sixth and last lines of (F5), that Carnot-engine DP power output is equal to (EBR-frictional retarding force) $\times$ (DP velocity) — i.e., that it is just sufficient to compensate for EBR-frictional drag on the DP — with frictional dissipation being recycled as per the last two sentences of the paragraph containing (29).

Maximizing $A[1 - (2e^{-A} - 1)^N]/(e^A - 1) \longrightarrow 1$ in (F5) as per the first sentence of the paragraph containing (30) and (31) (but within "practical" limits as per the fifth paragraph of Sect. 4 and as per Appendix B), applying (30) and the paragraph immediately following (24) in (F6), and then applying (38) and (F6) in (F7); Eqs. (C3) and (C4), respectively, of Ref. [3k] become

$$
\begin{aligned}
&\langle P^*_{\text{Carnot}}\rangle_{N,\max} = \langle P^*_{\text{Carnot}}\rangle_\infty |(A \longrightarrow 0) = \frac{64}{9}\sigma a T^4 \frac{\langle V\rangle^2_\infty |(A \longrightarrow 0)}{c^2} \\
&= \frac{64}{9}\sigma a T^4 \frac{(2V^2/3c)^2}{c^2} = \frac{256}{81}\sigma a T^4 \frac{V^4}{c^4} \\
&\implies \langle\langle P^*_{\text{Carnot}}\rangle\rangle_{N,\max} = \langle\langle P^*_{\text{Carnot}}\rangle\rangle_\infty |(A \longrightarrow 0) = \frac{64}{9}\sigma a T^4 \frac{\langle\langle V\rangle^2\rangle_{N,\max}}{c^2} \\
&= \frac{256}{81}\sigma a T^4 \frac{\langle\langle V^4\rangle\rangle_{\text{mw}}}{c^4} = \frac{256}{27}\sigma a T^4 \frac{(kT/M)^2}{c^4} = \frac{256}{27}\sigma a T^4 \frac{\langle\langle V^2\rangle\rangle^2_{\text{mw}}}{c^4}
\end{aligned} \tag{F6}
$$

and

$$
\mathcal{R}_{\text{I}} \equiv \frac{\langle\langle P^*\rangle\rangle_{\frac{1}{2},\max}}{\langle\langle P^*_{\text{Carnot}}\rangle\rangle_{N,\max}} = \frac{\langle\langle P^*\rangle\rangle_{\frac{1}{2}} |(A = 1)}{\langle\langle P^*_{\text{Carnot}}\rangle\rangle_\infty |(A \longrightarrow 0)} = \left(\frac{kM}{8T^7}\right)^{1/2} \frac{3c^2}{e^2 \sigma a L}. \tag{F7}
$$

For all reasonable values of M, T, a, and L in (F7), $\mathcal{R}_{\text{I}} \gg 1$ — but *also* with *qualitative* differences between the time evolutions of $\langle P^*\rangle_{N+\frac{1}{2}}$ of (22) and $\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}}$ of (28) on the one hand, and $\langle P^*_{\text{Carnot}}\rangle_N$ and $\langle\langle P^*_{\text{Carnot}}\rangle\rangle_N$ of (F5) on the other, as per the contrast between: (a) the pertinent discussions in Sects. 2 and 4 [especially recalling the two paragraphs immediately following that containing (32)] on the one hand, and (b) Case I of this Appendix F on the other. Of course, as per Sect. 3, the ratio $\mathcal{R}_{\text{I}}$ of (F7) also applies insofar as negentropy production is concerned. [Obviously, if, as per the last sentence of the paragraph containing (38), and the paragraph containing (39), $\langle P^*\rangle_{N+\frac{1}{2},\max} > \langle P^*\rangle_{\frac{1}{2},\max} \implies \langle\langle P^*\rangle\rangle_{N+\frac{1}{2},\max} > \langle\langle P^*\rangle\rangle_{\frac{1}{2},\max}$ for any $N \geq 1$, then $\mathcal{R}_{\text{I}}$ is also larger in the same proportion.]

Appendix F, Case II:

Given *fast* Carnot cycles (as per the second paragraph of this Appendix F), our DP Carnot engine operates "forwards" when $V > 0$ and "backwards" when $V < 0$ — and our interest is in its *net ("forwards")* operation. The referring to, in Appendix C [17] of Ref. [3k], of the DP's Carnot-engine power output as a "secondary" power output may be appropriate for Case I of this Appendix F — but it is a misnomer for Case II thereof: Said Case-II (as opposed to Case-I) Carnot-engine DP power output need not be smaller than the DP's so-called "primary" power output as per (28) and (38).

The temperature difference between the $+X$ and $-X$ disk faces corresponding to V is:

$$T_+(V) - T_-(V) = T\left(1 + \frac{2V}{3c}\right) - T\left(1 - \frac{2V}{3c}\right) = 4TV/3c \tag{F8}$$

— as per (2), the two sentences immediately following (2), and the paragraph immediately thereafter.

In numerical correction of the first paragraph of Appendix B of Ref. [3k] [including Eqs. (B1) and (B2) thereof]: Corresponding to V, the radiant power $r_\pm$ and radiant force $\mathcal{F}_\pm$ impinging on the $\pm X$ disk face — and thence $r_+ - r_-$ and $\mathcal{F}_+ - \mathcal{F}_-$, respectively — is, upon applying (2) and (F8), and letting σ be the Stefan-Boltzmann constant and $a =$ (cross-sectional area of disk) $=$ [$\pi\times$ (radius of disk)^2 $-$ (cross-sectional area of guide)],

$$\begin{aligned} r_\pm &= \sigma a T_\pm^4 = \sigma a \left[T\left(1 \pm \frac{2V}{3c}\right)\right]^4 = \sigma a T^4\left(1 \pm \frac{8V}{3c}\right) \\ \Longrightarrow\ r_+ - r_- &= \frac{16V}{3c}\sigma a T^4 \\ \Longrightarrow\ \mathcal{F}_\pm &= -\frac{4}{3}\left(1 \pm \frac{V}{3c}\right)\frac{r_\pm}{c} = -\frac{4}{3}\frac{\sigma a T^4}{c}\left(1 \pm \frac{3V}{c}\right) \\ \Longrightarrow\ \mathcal{F}_+ - \mathcal{F}_- \equiv \mathcal{F}_{\text{net}} &= -\frac{3}{2}\frac{r_+ - r_-}{c} = -\frac{8V}{c^2}\sigma a T^4. \end{aligned} \tag{F9}$$

The minus sign obtains in $\mathcal{F}_{\text{net}}$ because $\mathcal{F}_{\text{net}}$ tends to decrease $|V|$. [The last two lines of (F9) are justified as per the paragraph containing (B1) and (B2).] Since, corresponding to V, the DP's X-directional momentum is $p = MV$, $\mathcal{F}_{\text{net}}$ tends to decrease $|V|$ appreciably in an e-folding time scale of

$$\Delta t' = \left|\frac{p}{\mathcal{F}_{\text{net}}}\right| = \frac{MV}{\frac{8V}{c^2}\sigma a T^4} = \frac{Mc^2}{8\sigma a T^4}, \tag{F10}$$

which is *in*dependent of V. [Of course, because fluctuations in V at TEQ are time-symmetrical, $|V|$ will (as per the fluctuation-dissipation theorem) sometimes increase — but typically only in a like e-building time scale of $\Delta t'$ [33].]

Note again (as in Case I of this Appendix F) that — as per (F8), (F9), and the associated discussions — the numerator and denominator of Eq. (B3) of Ref. [3k] both require numerical correction by the *identical* factor of $2/3$ — hence, the rightmost term thereof is *un*changed since these two numerical corrections therein cancel out.

Also note again (as in Case I of this Appendix F) — accounting for differences in notation [13a,34] — that $\mathcal{F}_{\text{net}}$ is $4/3$ times that derived by Peebles [13a,34]. This obtains because our $\pm X$ disk faces are *flat* $\pm X$-directional EBR receptors, whereas a *spherical* particle (as considered by Peebles [13a,34]) has *hemispherical* $\pm X$-directional EBR receptors. Since EBR impinges *perpendicularly* ($\alpha = 0 \iff \cos\alpha = 1$) *onto a sphere from any and all directions*, for a sphere's *hemispherical* $\pm X$-directional EBR receptors,

(2) is modified to

$$
\begin{aligned}
T_{\pm}(V) &= \langle T_{\pm}(V,\alpha)\rangle \\
&= \frac{\int_0^{\pi/2} T\left(1 \pm \frac{V\cos\alpha}{c}\right)\sin\alpha d\alpha}{\int_0^{\pi/2}\sin\alpha d\alpha} \\
&= T\left(1 \pm \frac{V}{2c}\right). \qquad \text{(F11)}
\end{aligned}
$$

Hence, $T_+(V)-T_-(V)=V/c$ for *hemispherical* $\pm X$-directional EBR receptors — as opposed to $T_+(V)-T_-(V)=4V/3c$ for *flat* $\pm X$-directional EBR receptors. Since $\mathcal{F}_{\rm net} \propto T_+(V)-T_-(V)$, said factor of $4/3$ is explained.

It should be pointed out that said factor of $4/3$ obtains *only* given that — as in our system — the DP is constrained to *one*-dimensional (X-directional) motion by the guide. If it were *not* so constrained, so that it could manifest rotational as well as translational Brownian motion, then it would present a spherical cross-section on the average, and hence said factor of $4/3$ would *not* obtain. [Of course, $\operatorname{sgn}(T_+-T_-)$ would then *not* be correlated with the direction of DP motion, thus contravening our challenge to the second law.]

Applying (F8) and (F9), corresponding to V, the DP's Carnot-engine power output — assuming Carnot efficiency — is

$$
\begin{aligned}
P^*_{\rm Carnot}(V) &= r_+ - r_- \times \text{ (Carnot efficiency corresponding to } V) \\
&= r_+ - r_- \times \frac{T_+(V)-T_-(V)}{T_+(V)} \doteq r_+ - r_- \times \frac{T_+(V)-T_-(V)}{T} \\
&= \frac{16V}{3c}\sigma a T^4 \times \frac{4TV/3c}{T} = \frac{64}{9}\sigma a T^4 \frac{V^2}{c^2} \\
&= \left|\mathcal{F}_{\rm net} V\right|. \qquad \text{(F12)}
\end{aligned}
$$

The second step in the second line of (F12) is justified because V is nonrelativistic, with $|V| \ll c$ for all values of $|V|$ that have nonnegligible probabilities of being equaled or exceeded.

Again, given *fast* Carnot cycles (as per the second paragraph of this Appendix F), our DP Carnot engine operates "forwards" when $V > 0$ and "backwards" when $V < 0$ — and our interest is in its *net* *("forwards")* operation — i.e., in its *net* (*uni*-directional) Carnot-engine DP power output $=$ {[*net* (*uni*-directional: $+X \longrightarrow -X$ directional) heat flow through the DP] $\times$ (Carnot efficiency corresponding to V)

averaged over any *one* given $\pm|V|$ pair and thence over *all* $|V|\}$. Therefore, we have

$$\begin{aligned}
\left\langle P^*_{\text{Carnot,net}}\right\rangle_N &= P^*_{\text{Carnot}}(+|V|)P(+)_N - P^*_{\text{Carnot}}(-|V|)P(-)_N \\
&= P^*_{\text{Carnot}}(|V|)P(+)_N - P^*_{\text{Carnot}}(|V|)P(-)_N \\
&= P^*_{\text{Carnot}}(|V|)\left[P(+)_N - P(-)_N\right] \\
&= P^*_{\text{Carnot}}(|V|)\left\{P(+)_N - [1 - P(+)_N]\right\} \\
&= P^*_{\text{Carnot}}(|V|)\left[2P(+)_N - 1\right] \\
&= P^*_{\text{Carnot}}(|V|)\frac{(F-R)[1-(F+R-1)^N]}{2-F-R} \\
&= P^*_{\text{Carnot}}(|V|)\frac{(4|V|/3c)\,Ae^{-A}\left[1-\left(2e^{-A}-1\right)^N\right]}{2-2e^{-A}} \\
&= P^*_{\text{Carnot}}(|V|)\frac{2|V|}{3c}\frac{A\left[1-\left(2e^{-A}-1\right)^N\right]}{e^A-1} \\
&= \frac{64}{9}\sigma a T^4\frac{V^2}{c^2}\times\frac{2|V|}{3c}\frac{A\left[1-\left(2e^{-A}-1\right)^N\right]}{e^A-1} \\
&= \frac{128}{27}\sigma a T^4\frac{|V|^3}{c^3}\frac{A\left[1-\left(2e^{-A}-1\right)^N\right]}{e^A-1} \\
&= \langle|\mathcal{F}_{\text{net}}V|\rangle_N \\
\Longrightarrow\ \left\langle\left\langle P^*_{\text{Carnot,net}}\right\rangle\right\rangle_N &= \frac{128}{27}\sigma a T^4\frac{\left\langle\left\langle|V|^3\right\rangle\right\rangle_{\text{mw}}}{c^3}\frac{A\left[1-\left(2e^{-A}-1\right)^N\right]}{e^A-1} \\
&= \frac{128}{27}\sigma a T^4\frac{\left[2\left(kT/M\right)^3\right]^{1/2}}{c^3}\frac{A\left[1-\left(2e^{-A}-1\right)^N\right]}{e^A-1} \\
&= \langle\langle|\mathcal{F}_{\text{net}}V|\rangle\rangle_N\,.
\end{aligned} \tag{F13}$$

In the second step of (F13), we applied (F12); in the fourth step, $P(+)_N + P(-)N = 1$; in the sixth step, the last line of (10); in the seventh step, (5) and (6); and in the ninth step, (F12) again. Via comparison of (11), (18), (19), (22), (25), and (28) on the one hand with (F13) on the other, time evolution and maximization of $\left\langle P^*_{\text{Carnot,net}}\right\rangle_N$ ($\left\langle\left\langle P^*_{\text{Carnot,net}}\right\rangle\right\rangle_N$) is identical with that of $\langle V\rangle_N$ ($\langle\langle V\rangle\rangle_N$), rather than with that of than of $\langle P^*\rangle_N$ ($\langle\langle P^*\rangle\rangle_N$).

Note, as per the last line of (F12) and the third-to-the-last and last lines of (F13), that [similarly to Case I as per the sixth, seventh, and last lines of (F5) and the second paragraph following (F5)], Case-II Carnot-engine DP power output is equal to (EBR-frictional retarding force) × (DP velocity) — i.e., that it is just sufficient to compensate for EBR-frictional drag on the DP — with frictional dissipation being recycled as per the last two sentences of the paragraph containing (29). But there are two *dis*similarities: (a) the Case-I products of averages of $\langle|f_{\text{net}}|\rangle_N\,\langle V\rangle_N$ and $\langle\langle|f_{\text{net}}|\rangle\rangle_N\,\langle\langle V\rangle\rangle_N$, as opposed to the Case-II averages of the products $\langle|f_{\text{net}}|\,V\rangle_N$ and $\langle\langle|f_{\text{net}}|\,V\rangle\rangle_N$; and (b) Case-II DP Carnot-engine power outputs and frictional-drag losses are both typically larger — by the same ratio — than their Case-I counterparts.

Maximizing $A[1-(2e^{-A}-1)^N]/(e^A-1) \longrightarrow 1$ in (F13) as per the first sentence of the paragraph containing (30) and (31) (but within "practical" limits as per the fifth paragraph of Sect. 4 and as per Appendix B), applying (30) and the paragraph immediately following (24) in (F14), and then applying (38) and (F14) in (F15); the Case-II analogs of Eqs. (C3) and (C4), respectively, of Ref. [3k] are

$$\begin{aligned} \left\langle P^*_{\text{Carnot,net}} \right\rangle_{\max} &= \left\langle P^*_{\text{Carnot,net}} \right\rangle_{\infty} |(A \longrightarrow 0) = \frac{128}{27} \sigma a T^4 \frac{|V|^3}{c^3} \\ \Longrightarrow \left\langle\left\langle P^*_{\text{Carnot,net}} \right\rangle\right\rangle_{\max} &= \left\langle\left\langle P^*_{\text{Carnot,net}} \right\rangle\right\rangle_{\infty} |(A \longrightarrow 0) = \frac{128}{27} \sigma a T^4 \frac{\left\langle\left\langle |V|^3 \right\rangle\right\rangle_{\text{mw}}}{c^3} \\ &= \frac{128}{27} \sigma a T^4 \frac{\left[2\left(kT/M\right)^3\right]^{1/2}}{c^3} \end{aligned} \tag{F14}$$

and

$$\mathcal{R}_{\text{II}} \equiv \frac{\langle\langle P^* \rangle\rangle_{\frac{1}{2},\max}}{\left\langle\left\langle P^*_{\text{Carnot,net}} \right\rangle\right\rangle_{N,\max}} = \frac{\langle\langle P^* \rangle\rangle_{\frac{1}{2}} |(A = 1)}{\left\langle\left\langle P^*_{\text{Carnot,net}} \right\rangle\right\rangle_{\infty} |(A \longrightarrow 0)} = \frac{3kc}{2e^2 L \sigma a T^3}. \tag{F15}$$

For reasonable values of M, T, a, and L in (F15), $\mathcal{R}_{\text{II}}$ may be within a few orders of magnitude of unity ($R_{\text{II}} = 1$ if $T = 300\,\text{K}$ and $La \doteq 5.49 \times 10^{-16}\,\text{m}^3$). Note the *qualitative* differences between the time evolutions of (a) $\langle P^* \rangle_{N+\frac{1}{2}}$ of (22) and $\langle\langle P^* \rangle\rangle_{N+\frac{1}{2}}$ of (28), (b) $\langle P^*_{\text{Carnot}} \rangle_N$ and $\langle\langle P^*_{\text{Carnot}} \rangle\rangle_N$ of (F5), and (c) $\left\langle P^*_{\text{Carnot,net}} \right\rangle_N$ and $\left\langle\left\langle P^*_{\text{Carnot,net}} \right\rangle\right\rangle_N$ of (F13), as per the contrast between the pertinent discussions in Sects. 2 and 4 [especially recalling the two paragraphs immediately following that containing (32)] on the one hand, and Cases I and II of this Appendix F on the other. Of course, as per Sect. 3, the ratio $\mathcal{R}_{\text{II}}$ of (F15) also applies insofar as negentropy production is concerned. [Obviously, if, as per the last sentence of the paragraph containing (38), and the paragraph containing (39), $\langle P^* \rangle_{N+\frac{1}{2},\max} > \langle P^* \rangle_{\frac{1}{2},\max} \implies \langle\langle P^* \rangle\rangle_{N+\frac{1}{2},\max} > \langle\langle P^* \rangle\rangle_{\frac{1}{2},\max}$ for any $N \geq 1$, then $\mathcal{R}_{\text{II}}$ of (F15) is also larger in the same proportion.]

Appendix G: DP with a nonrelativistic thermal background medium

We consider a DP for which a nonrelativistic positive-rest-mass gas thermal background medium at temperature T is preponderant over the EBR [25], so that $c \longrightarrow |U|$, with $|U|$ on the order of a typical thermal or sonic molecular speed in said medium, rather than the speed of light in vacuum — yielding the advantage of $|U| \ll c \implies |V|/|U| \gg |V|/c$ [25]. A further advantage obtains if DP size and mass given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR can be smaller than those given the EBR being the *sole* thermal background medium. (Even given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR, there seems to be no advantage in "excluding" the EBR corresponding to T: such "exclusion" begins to obtain if the DP is enclosed within, say, a conducting shell of diameter $\lesssim hc/kT \approx$ wavelength of typical EBR photon at temperature T, and obtains *strongly* if said diameter $\ll hc/kT$.)

Both V and U are relative to the reference frame of our system wherein the guide and peg row are at rest (recall Fig. 1 and the two immediately following paragraphs, especially the second-to-the-last sentence of the former paragraph).

The form of a Planck distribution survives Doppler shifting, but that of a Maxwellian distribution does not [13,25]. Hence — while both Planck and Maxwellian distributions rigorously have temperatures if not

Doppler-shifted — a Doppler-shifted Planck distribution rigorously has a temperature, whereas a Doppler-shifted Maxwellian distribution does not [13,25]. Nevertheless, if the kinetic energy of nonrelativistic gas molecules (for simplicity, each taken to be of identical mass $m_{\rm gas}$) is *rapidly thermalized in the DP's instantaneous rest frame* when they strike the DP, then (1) – (6) and the associated discussions, and hence the essential features of our analyses, still obtain *upon thermalization*, provided that (taking U to be the thermal velocity of a gas molecule) we let $c \longrightarrow |U|$ [25]. However, in order for the DP to operate, the equivalent of $\Delta t'' \ll \Delta t'$ as per Appendix B must also still obtain. We now derive the requirement for this to be the case.

For simplicity, let us assume a given initial V for our DP, and assume that all gas molecules have the same speed $|U|$, but with randomly distributed directions of motion of the gas molecules relative to the reference frame of our system wherein the guide and peg row are at rest (recall Fig. 1 and the two immediately following paragraphs, especially the second-to-the-last sentence of the former paragraph). This assumption will turn out to be justified, because our final result of this Appendix G will turn out to be (for all practical purposes) *in*dependent of V and $|U|$ — hence, averaging over V and $|U|$ will turn out to be (for all practical purposes) *un*necessary. (Note: For brevity of notation, we do *not* denote such averages as do occur in this Appendix G via enclosure within angular brackets.) Also, for simplicity, we assume that $M \gg m_{\rm gas} \implies |V| \ll |U|$ for thermal (Brownian) $|V|$ and $|U|$ almost always. Let $a =$ (cross-sectional area of disk) $= [\pi\times$ (radius of disk)$^2 -$ (cross-sectional area of guide)], and let ρ be the number density of gas molecules per unit volume. Let l be the average X-directional spatial separation between two gas molecules that — *within the "sweep"* of the cross-sectional area a of the disk — are adjacent (with no intervening molecules) in the X direction (although other molecules within its "sweep" may be spatially closer to either or both of said two molecules in the Y and/or Z directions). Then $l = 1/\rho a$ [for simplicity, we take $l \gg$ (gas-molecule diameter)]. [Our final result of this Appendix G will turn out to be *in*dependent of l, ρ, a, M, $m_{\rm gas}$, and T— as well (for all practical purposes) of V and $|U|$!] Averaging over all angles of impingement, gas molecules strike the $+X$ $[-X]$ disk face with an average X-directional component of velocity relative to the disk's instantaneous rest frame of $U^+_{\rm rel} = -V - \frac{2|U|}{3}$ $[U^-_{\rm rel} = -V + \frac{2|U|}{3}]$; the factor of $\frac{2}{3}$ arising similarly as in (2). The average time interval between impingements of gas molecules on the $+X$ $[-X]$ disk face is $\Delta t^+ = 2l\,/\left|U^+_{\rm rel}\right| = 2l\left/\left|-V - \frac{2|U|}{3}\right|\right. = 2l\left/\left(\frac{2|U|}{3} + V\right)\right.$ $\left[\Delta t^- = 2l\,/\left|U^-_{\rm rel}\right| = 2l\left/\left|-V + \frac{2|U|}{3}\right|\right. = 2l\left/\left(\frac{2|U|}{3} - V\right)\right.\right.$ because $\frac{2|U|}{3} > |V|$ for thermal (Brownian) $|V|$ and $|U|$ almost always]; with $2l$ rather than l in the numerators of $\Delta t^{\pm}$ because, on the average, *half* the gas molecules move in either the $+X$ or $-X$ direction. An average gas molecule impinging on *and thermalizing with* the $+X$ $[-X]$ disk face rebounds at $U^+_{\rm rel} = \frac{2|U|}{3}$ $[U^-_{\rm rel} = -\frac{2|U|}{3}]$ — and hence (a) suffers $\Delta U^+_{\rm rel} = V + \frac{4|U|}{3}$ $[\Delta U^-_{\rm rel} = V - \frac{4|U|}{3}]$ and (b) by Newton's third law, imparts onto the $+X$ $[-X]$ disk face a momentum change of $\Delta p^+ = -m_{\rm gas}\Delta U^+_{\rm rel} = -m_{\rm gas}(V + \frac{4|U|}{3})$ $[\Delta p^- = -m_{\rm gas}\Delta U^-_{\rm rel} = -m_{\rm gas}(V - \frac{4|U|}{3})]$. By Newton's second law, the force thus exerted on the $+X$ $[-X]$ disk face is that given the $+$ $[-]$ signs of the $\pm$ symbols in (G1):

$$\begin{aligned} f^{\pm} &= \Delta p^{\pm}/\Delta t^{\pm} = -m_{\text{gas}}\Delta U^{\pm}_{\text{rel}}/\Delta t^{\pm} \\ &= -m_{\text{gas}}\left(V \pm \frac{4\,|U|}{3}\right) \div \frac{2l}{\frac{2|U|}{3} \pm V} \\ &= -\frac{m_{\text{gas}}}{2l}\left(V \pm \frac{4\,|U|}{3}\right)\left(\frac{2\,|U|}{3} \pm V\right) \\ &= -\frac{m_{\text{gas}}}{2l}\left(\pm V^2 + 2V\,|U| \pm \frac{8}{9}U^2\right). \end{aligned} \tag{G1}$$

The *net* force exerted on the DP is therefore

$$f_{\text{net}} = f^{+} + f^{-} = -\frac{m_{\text{gas}}}{2l}(4V\,|U|) = -\frac{2m_{\text{gas}}V\,|U|}{l}. \tag{G2}$$

The DP's (X-directional) momentum is MV, hence, V is changed appreciably compared with its initial value in an e-folding time scale of

$$\Delta t' = M\,|V/f_{\text{net}}| = MV \div \frac{2m_{\text{gas}}V\,|U|}{l} = \frac{Ml}{2m_{\text{gas}}\,|U|}, \tag{G3}$$

which is *in*dependent of V (as with EBR as the thermal background medium, as per Appendix B).

Recalling that $\Delta t^{\pm} = 2l \Big/ \left|U^{\pm}_{\text{rel}}\right| = 2l \Big/ \left(\frac{2|U|}{3} \pm V\right)$ and assuming that $|V| \ll \frac{2|U|}{3}$, during $\Delta t'$, each disk face suffers approximately $\mathcal{N} = \frac{\Delta t'}{\Delta t^{\pm}} \approx \frac{Ml}{2m_{\text{gas}}|U|} \div \frac{2l}{2|U|/3} = \frac{M}{6m_{\text{gas}}}$ gas-molecule impacts. These impacts tend to decrease $|V|$ towards 0 [although, being random, they can (as per the fluctuation-dissipation theorem) also increase $|V|$ — typically in a like e-building time scale of $\Delta t'$] [33]. Each impact is of energy $\approx kT$.

For argument's sake, take an initial value $V > 0$, and consider the $+X$ disk *half*. During $\Delta t'$, as said initial velocity tends to be reduced towards 0, the average value is $\approx \frac{1}{2}V$. Assuming *rapid thermalization* at gas-molecule impacts on the DP, corresponding to $\frac{1}{2}V$, the temperature of the $+X$ disk face is $T(1+\frac{1}{2}\frac{2V}{3|U|}) = T(1+\frac{V}{3|U|})$. Thus, per gas-molecule impact on the $+X$ disk face, $\approx \frac{V}{3|U|}kT$ of thermal energy *is* transferred from the $+X$ disk face to the gas; and, for all $\mathcal{N}$ impacts during $\Delta t'$, $\epsilon' \approx \mathcal{N}\frac{V}{3|U|}kT \approx \frac{M}{6m_{\text{gas}}}\frac{V}{3|U|}kT = \frac{M}{18m_{\text{gas}}}\frac{V}{|U|}kT$ *is* transferred. The thermal energy *required* to be transferred to the gas from the $+X$ disk *half* during $\Delta t'$ to reduce the latter's spatially-averaged temperature from $T(1+\frac{1}{2}\frac{2V}{3|U|}) = T(1+\frac{V}{3|U|})$ (assuming, for simplicity, a uniform X-directional temperature gradient within the disk corresponding to the initial V) to T (corresponding to $V = 0$) is $\epsilon'' = \frac{1}{2}MC^{*}T\frac{V}{3|U|} = \frac{1}{6}MC^{*}T\frac{V}{|U|}$, where C^{*} is the DP's specific heat per unit mass. The requirement equivalent to $\Delta t'' \ll \Delta t'$ of Appendix B is thus

$$\epsilon'' \ll \epsilon' \implies \frac{1}{6}MC^{*}T\frac{V}{|U|} \ll \frac{M}{18m_{\text{gas}}}\frac{V}{|U|}kT \implies C^{*} \ll \frac{k}{3m_{\text{gas}}}. \tag{G4}$$

Note that this result is *in*dependent of l, ρ, a, M, m_{gas}, and T— as well (for all practical purposes) of V and $|U|$. Moreover, letting $\mathfrak{N}$ be the number of atoms comprising the DP (for simplicity, each taken to be of identical mass m_{DP}),

$$C^{*} = \frac{\mathfrak{N}\phi k}{M} = \frac{\mathfrak{N}\phi k}{\mathfrak{N}m_{\text{DP}}} = \frac{\phi k}{m_{\text{DP}}}. \tag{G5}$$

ϕk is the specific heat per DP atom: classically, typically, $\phi \approx 1$; quantum-mechanically, $0 < \phi \lesssim 1$; $\phi \gg 1$ rarely obtains [37]. Applying (G5) to (G4) yields our final result of this Appendix G [which is *in*dependent of l, ρ, a, M, $\mathfrak{N}$, $m_{\rm DP}$, $m_{\rm gas}$, and T — as well (for all practical purposes) of V and $|U|$]:

$$\begin{aligned} \epsilon'' \ll \epsilon' &\implies C^* \ll \frac{k}{3m_{\rm gas}} \\ &\implies \frac{\phi k}{m_{\rm DP}} \ll \frac{k}{3m_{\rm gas}} \\ &\implies \phi \ll \frac{m_{\rm DP}}{3m_{\rm gas}}. \end{aligned} \tag{G6}$$

Said final result (G6) — which is more complete than our previous statement (according to which $\phi \ll k$ [38]) — is easily realizable (at least in principle), since $m_{\rm DP} \ll m_{\rm gas} \ll M = \mathfrak{N} m_{\rm DP}$ can easily obtain: e.g., the gas molecules could be (multi-atomic, yet still small) Brownian particles. Since, for any given L (and, assuming uniform scaling, for any given $M \propto L^3$), $\langle\langle V\rangle\rangle_N$ can be improved by a ratio of $c/\langle\langle|U|\rangle\rangle_{\rm mw}$ — and $\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}}$, the time rate of negentropy production, and corresponding densities thereof by a ratio of $(c/(\langle\langle|U|\rangle\rangle_{\rm mw})^2$ — perhaps this is worth considering [39]. {So long as $|V| \ll |U|$ almost always, and $|U| \ll c$ for all $|U|$ that have nonnegligible probabilities of being equaled or exceeded, the angular dependencies [as per the paragraph containing (1) and (2)] are the same for photons and gas molecules and therefore "cancel out": hence, the numerical factors *not* equal to unity in Ref. [39] are in error.}

Note that, as in the case of the EBR thermal background medium, $\langle\langle P^*\rangle\rangle_{N+\frac{1}{2}}$ and time rate of negentropy production scale as $L^{-11/2}$, and densities thereof as $L^{-17/2}$. But, in contrast with the EBR case, scaling with temperature is as $T^{3/2}$ — not as $T^{5/2}$ — because, given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR, $c^2 \longrightarrow U^2 \approx \langle\langle|U|\rangle\rangle^2_{\rm mw} \propto T$ [39] in the denominators of (28) and (38) [with $c \longrightarrow U$ in (28) and (38)].

A further advantage obtains if DP size and mass given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR can be smaller than those given the EBR being the *sole* thermal background medium: This seems reasonable because, for any given T, the typical thermal deBroglie wavelength of (nonrelativistic) gas molecules, i.e., $h/m\,\langle\langle|U|\rangle\rangle_{\rm mw} \approx h/(mkT)^{1/2}$, as well as the physical size of said gas molecules, typically is much smaller than $\approx hc/kT$, the wavelength of typical EBR photons at temperature T.

Hence, perhaps, issues such as those discussed in the second and third paragraphs of Sect. 5 [19–21] might be more easily circumvented — or simply be rendered unimportant — given a nonrelativistic positive-rest-mass thermal background medium (e.g., our gas) being preponderant over the EBR, as opposed to the EBR being the sole thermal background medium [as per Item (c) in the last paragraph of Sect. 5] [22–25]. (Even given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR, there seems to be no advantage in "excluding" the EBR corresponding to T: such "exclusion" begins to obtain if the DP is enclosed within, say, a conducting shell of diameter $\lesssim hc/kT \approx$ wavelength of typical EBR photon at temperature T, and obtains *strongly* if said diameter $\ll hc/kT$.)

Additionally, note that the DP's Carnot-engine power outputs as per Appendix F also may be improved given a nonrelativistic positive-rest-mass thermal background medium being preponderant over the EBR, because $T_+(V) - T_-(V)$ is then $\propto V/|U|$ rather than $\propto V/c$ [but with $\mathcal{R}_{\rm I}$ of (F7) and $\mathcal{R}_{\rm II}$ of (F15) remaining unchanged].

Appendix H: Friction and stiction

Powered nanotechnological devices, which typically are *driven* by negentropy (and hence free-energy) expenditure at speeds far exceeding those typical of their Brownian motions, often "seize up" under the action of friction, and especially of stiction [40]. However, *un*powered nanotechnological devices, whose *only* motions are Brownian (even if the *randomness* of this Brownian motion is spontaneously broken in challenge of the second law as in the case of our DP) can*not* so "seize up". The fluctuation-dissipation theorem forbids [33]! The fluctuation-dissipation theorem requires that, for *un*powered devices, dissipation of Brownian motion must be matched by its generation: a strong dissipative effect such as stiction must also be an equally strong generative effect. Strong dissipative-generative effects result in Brownian-motional velocity V fluctuating more rapidly than weak ones; but for any *un*powered device (with given T/M), $\langle\langle|V|\rangle\rangle_{\mathrm{mw}}$ is identical irrespective of whether said effects are strong or weak: e.g., "seizing" and "antiseizing" are equally probable. Only for typical *powered* nanotechnological devices is friction, and especially stiction, much more likely to cause "seizing" than "antiseizing" — because *powered* nanotechnological devices typically are *driven* at $|V| \gg \langle\langle|V|\rangle\rangle_{\mathrm{mw}}$. Note, however, that *powered* nanotechnological devices *can* be *driven* at *any non*-Brownian motion: For example, if *driven (frozen?)* at $|V| \ll \langle\langle|V|\rangle\rangle_{\mathrm{mw}}$, then they are, as per the fluctuation-dissipation theorem [33], subject to "antiseizing" much more than to "seizing"!

As per the immediately preceding paragraph, the fluctuation-dissipation theorem [33] suffices to preserve average Brownian-motional (thermal) *speed* — which is all that is required insofar as the *pawl's* average Z-directional Brownian-motional (thermal) *speed* $\langle|v|\rangle_{\mathrm{mw}}$ is concerned. However, if DP operation is to challenge the second law, then it is, of course, *also* necessary to preserve the *DP's* X-directional Brownian-motional systematic net drift *velocity* $\langle V\rangle_N > 0 \implies \langle\langle V\rangle\rangle_N > 0$ (given $|V| > 0$ and $N \geq 1$) — *not* merely its average X-directional Brownian-motional (thermal) *speed* $\langle\langle|V|\rangle\rangle_{\mathrm{mw}} = (2kT/\pi M)^{1/2}$. If strong dissipative-generative effects such as stiction cause n reversals in $\operatorname{sgn} V$ between consecutive pawl-peg interactions, then said drift *velocity* will, by the laws of statistics, be reduced $n^{1/2}$-fold, with corresponding reductions in accelerating force, power output, and negentropy production.

Perhaps, for our DP, friction is modeled, at least to a first approximation, as per Appendix G: Since our final result (G6) of Appendix G is *in*dependent of gas density, perhaps, at least as a first approximation, it is *in*dependent of density in general — i.e., perhaps, at least as a first approximation, it is correct if, say, our guide (recall Fig. 1 and the immediately following paragraph) is *not* frictionless. If (G6) can thus — even if only as a first approximation — be correct (even considering friction), then, perhaps, there can be, on the average, much less than one reversal of $\operatorname{sgn} V$ between consecutive pawl-peg interactions. But this still seems to be an open question.

Appendix I: Some comparisons, clarifications, and improvements concerning Ref. [3k]

The dynamically correct quantitative time-evolutionary treatment developed in this present paper confirms the main results of Ref. [3k], which could be derived only semiquantitatively and/or stated only incompletely in Ref. [3k]. Reference [3k] lacked said time-evolutionary treatment, which is required for the more complete understanding of DP operation that is provided by this present paper. Yet Ref. [3k] still showed that velocity-dependence of fluctuations can spontaneously break the randomness of Brownian motion at TEQ — hence challenging the second law. Also, most of the results that *were* obtained in Ref. [3k], e.g., for temperature of Doppler-shifted EBR, systematic net drift velocity, power output, and a few other quantities, were semiquantitatively correct (of correct functional form, i.e., in error only numerically) — despite their derivations in Ref. [3k] not being entirely correct. Nevertheless, many results derived in this present paper could not be derived — not even semiquantitatively — in Ref. [3k].

Examples of conceptual and notational improvements, as well as of numerical corrections, should be noted, so that Ref. [3k] can be referred to more consistently with respect to this present paper. Many of said improvements and corrections have also been incorporated into previous shorter versions [28] of this present paper.

In Ref. [3k], our DP is referred to as our ratchet and pawl (RP). In this present paper [as per the third paragraph of Sect. 2 and the paragraph immediately following (24) thereof], but not in Ref. [3k], if a quantity Q or an average thereof is time-dependent, then its value at time N is indicated via a subscript N; with $\langle Q\rangle_N$ ($\langle\langle Q\rangle\rangle_N$) denoting the expectation value at time N of a quantity Q over any *one* given $\pm|V|$ pair ($\langle Q\rangle_N$ *itself* averaged over *all* $|V|$). Corresponding (Ref. [3k] $\longrightarrow$ this present paper) notational improvements include $V_{\rm net}(V) \longrightarrow \langle V\rangle_N$ and $V_{\rm net} \longrightarrow \langle\langle V\rangle\rangle_N$. The subscript "opt" (for "optimum") is used in this present paper *only* to denote our *primary* objective of maximum power output and hence maximum time rate of negentropy production — not (as in Ref. [3k]) *also* to denote maximum systematic net drift velocity. The result as per the last three paragraphs of Sect. IIC and the first two paragraphs of Sect. IID of Ref. [3k] that the value $A = 1$ corresponds both to maximum systematic net drift velocity and to maximum power output has been superseded in this present paper.

The optimum slope angle $\theta_{\rm opt}$, corresponding to maximum power output and hence to maximum time rate of negentropy production, as per the sentence immediately following (29) and the last sentence of the paragraph immediately following that containing (32) of this present paper, must, in general, be calculated numerically: the result obtained as per the second paragraph of Sect. IID of Ref. [3k] requires, in general, numerical correction.

The approximation made in Ref. [3k] that all EBR impinging on both $\pm X$ disk faces has *purely* X-directional motion has been superseded in this present paper by correctly (to first order in V/c) averaging over *all* impingement angles as per the paragraph containing (1) and (2) [13]. The resulting numerical correction is $V/c \longrightarrow 2V/3c$.

Conceptual and notational improvements, as well as numerical corrections, for Appendixes B and C of Ref. [3k] are as indicated in Appendixes B and F of this present paper.

The error, in the paragraph immediately following that containing Eq. (23) of Ref. [3k], concerning the effect of *diffraction* on the DP's EBR absorption/(re)radiation efficiency as a function of DP size if DP size $\lesssim hc/kT$ (and most strongly if DP size $\ll hc/kT$) [19], has been corrected [20] — and supplementary discussions given [20–24] — in the last three paragraphs of Sect. 5 of this present paper.

The statement in the third paragraph of Appendix D [25] of Ref. [3k] that, for rapid thermalization of a "gas DP", the specific heat per DP atom must be $\ll k$ [38], is incomplete; and is superseded by Appendix G — especially by the more complete final result (G6) thereof and the immediately following sentence — of this present paper.

The ratios given in the first paragraph of Appendix D [25] of Ref. [3k], immediately following Eq. (D1) thereof [39], should have no numerical factors (other than unity), as per the fifth- and fourth-to-the-last paragraph of Appendix G of this present paper.

Appendix J: Corrections for Ref. [28b]

The following corrections are required for Ref. [28b]:

Page 76.2: 13636 Neutron ... 75344-4410 $\longrightarrow$ 3233 West Kingsley Road; Garland, Texas 75041-2205
Page 76.2: elgt@elgt-amti.com $\longrightarrow$ info@elgt.com
Page 77.2: and free energy /// (and hence free energy)
Page 77, last line: $H_{\text{net}} = H - Z_{\text{min}} \longrightarrow H_{\text{net}} \equiv H - Z_{\text{min}}$
Page 78, first line: The DP, and the entire system, $\longrightarrow$ The entire system, including the DP,
Page 78.2: Insert as last sentence of first paragraph ending on Page 78: Except for the EBR, our system is nonrelativistic: i.e., all speeds (except of EBR photons) are $\ll c$, $gH \ll c^2$, and all pertinent differences in gravitational potential (e.g., gH) are $\ll c^2$.
Page 78.75: *itself* [11]. (See the Appendix.) $\longrightarrow$ *itself* [11] — *not* merely of Doppler-shifted EBR "seen" thereby [12]. (See the Appendix.)
Page 78.85: Replace the first step of Eq. (3) with the following two steps:

$$P(Z > H|V) = \exp[-mg(H - Z_{\text{min}})/kT_{+}(V)] \equiv \exp[-mgH_{\text{net}}/kT_{+}(V)]$$

Page 78.9, line immediately following Eq. (3): Insert: because $|V| \ll c \longrightarrow$ because V is nonrelativistic, with $|V| \ll c$
Page 79.15: $H_{\text{net}} = H - Z_{\text{min}} \longrightarrow H_{\text{net}} \equiv H - Z_{\text{min}}$
Page 79.15: L, g, m', m, $M = m' + m \gg m$, and $\longrightarrow$ L, m', m, $M = m' + m \gg m$, g, and
Page 79.7: *itself* averaged /// *itself* subsequently averaged
Page 81.6: nonzero rest-mass /// positive-rest-mass
Page 82.15 – 82.2 (3 occurrences): momentum exchanges $\longrightarrow$ momentum and kinetic-energy exchanges
Page 82.2: "sampling" /// sampling
Ref. [1a]: Ref. [1a] considers only velocity-dependent forces that are perpendicular to the velocity (i.e., to the direction of motion) $\longrightarrow$ Ref. [1a] considers *only* velocity-dependent forces that are perpendicular to the velocity *itself* (i.e., to the direction of motion)
Ref. [3a]: 46–x $\longrightarrow$ 46-x (x = 1, 2, 3, 4, 5)
Ref. [3d]: 6454 $\longrightarrow$ 6459
Ref. [3e]: Hernandez $\longrightarrow$ Hernández
Ref. [3k]: 5390-5399 $\longrightarrow$ 5390–5399
Ref. [3k]: 3390, Sect. I $\longrightarrow$ 3390. See Sect. I
Ref. [4a] (2 occurrences): *Proceedings 643* /// *Proceedings (Vol. 643)*
Ref. [4a]: AIP: New York, $\longrightarrow$ AIP: Melville, New York,
Ref. [5e]: Allahverdyan, A. E.; Nieuwenhuizen, Th. M. $\longrightarrow$ Nieuwenhuizen, Th. M.; Allahverdyan, A. E.
Ref. [5j]: Novotny $\longrightarrow$ Novotný
Ref. [6f]: Arsitov, A. A. $\longrightarrow$ Aristov, V. V.
Ref. [7]: 461-479 $\longrightarrow$ 461–479
Ref. [8c]: *(No.18)* $\longrightarrow$ *(No. 18)*
Ref. [11a]: Sect. IIA and the first two paragraphs of Sect. IIIC $\longrightarrow$ Sect. IIA, the first two paragraphs of Sect. IIC, and Sect. III
Ref. [11a]: in Appendixes A and C and in Footnote 7, $\longrightarrow$ in Appendixes A, B, and C, and in Footnote 7;

Ref. [11b]: **2004**. Regarding $\longrightarrow$ **2004**. Our system is described in Fig. 1 and the two immediately following paragraphs. Regarding
Ref. [11b]: Appendixes B, C, and I $\longrightarrow$ Appendixes B, C, F, and I
Ref. [11b]: Sect. V $\longrightarrow$ Sect. 5
Ref. [11c]: its surface area $\longrightarrow$ the pawl's undersurface area
Ref. [12b]: Misner, C. W., Kip S. Thorne, K. S. $\longrightarrow$ Misner, C. W.; Thorne, K. S.
Ref. [14b]: reissued: Waveland $\longrightarrow$ reissued by Waveland
Insert Ref. [14c]: (c) Ref. [11b], the paragraphs containing Eqs. (7) – (10).
Ref. [16]: Sect. IV $\longrightarrow$ Sect. 4
Footnote [22] (2 occurrences): *momentum* exchanges $\longrightarrow$ *momentum and kinetic-energy* exchanges
Footnote [23]: T, rather than $T_+(V)$, in $P(v)_{\mathrm{mw}} \longrightarrow$ the V-*in*dependent $P(v)_{\mathrm{mw}} = (m/2\pi kT)^{1/2} \times \exp(-mv^2/kT)$, rather than the V-*dependent* $P(v|V)_{\mathrm{mw}} = [m/2\pi kT_+(V)]^{1/2} \exp[-mv^2/kT_+(V)]$

Cited journal article titles (in the References and Notes) requiring correction of case (upper or lower) of first letter(s) of word(s) therein, with case corrected as necessary to that in the *actual* cited journal article titles: Ref. [1a]: Mechanical models of Maxwell's demon with noninvariant phase volume. Ref. [3e]: Feynman's ratchet optimization: maximum power and maximum efficiency regimes. Ref. [5h]: Extracting Work from a Single Thermal Bath via Quantum Negentropy. Ref. [5k]: Coherent Power Booster. Ref. [5l]: Extracting Work from a Single Heat Bath via Vanishing Quantum Coherence. Ref. [6e]: Observation of the External-ac-Current-Induced dc Voltage Proportional to the Steady Current in Superconducting Loops. Ref. [8b]: A Solid-State Maxwell Demon.

Finally, I overlooked deactivating the default setting (in Scientific WorkPlace 4.0 by MacKichan Software, Inc.) for the footer, resulting in an extra space of approximately $\frac{1}{2}\,\mathrm{in} = 1.27\,\mathrm{cm}$ at the bottoms of the pages, in excess of the $1.74\,\mathrm{cm}$ prescribed in Entropy's Instructions for Authors.

References and Notes

1. (a) Zhang, K.; Zhang, K. Mechanical models of Maxwell's demon with noninvariant phase volume. *Phys. Rev. A* **1992**, *46*, 4598–4605. [Reference [1a] is a classical rather than quantum-mechanical treatment; as is true of this present paper (except for the last four paragraphs of Sect. 3, Sect. 6, and a few very brief mentions elsewhere, therein), and also of Refs. [3k], [28a], and [28b]; but, in contrast therewith, Ref. [1a] considers only (*non*dissipative) velocity-dependent forces that are perpendicular to the velocity *itself* (i.e., to the direction of motion), and hence which can do no work (e.g., the Lorentz force)]. Reference [1a] was partially anticipated earlier by
(b) Bridgman, P. W. Note on the Principle of Detailed Balancing. *Phys. Rev.* **1928**, *31*, 101–102; and, in extension, by
(c) Davies, P. C. W. *The Physics of Time Asymmetry*. University of California Press: Berkeley, **1974**; Sect. 6.4 (especially the last paragraph).
Reference [1a] is summarized in
(d) Čápek, V.; Sheehan, D. P. *Challenges to the Second Law of Thermodynamics: Theory and Experiment*. Springer: Dordrecht, **2005** (Fundamental Theories of Physics Series, Vol. 146), Sect. 10.1.2.
2. Allahverdyan, A. E.; Nieuwenhuizen, Th. M. A mathematical theorem as the basis for the second law: Thomson's formulation applied to equilibrium. *Physica A* **2002**, *305*, 542–552. See the first and fifth paragraphs of Sect. 5.

3. (a) Feynman's classic ratchet and pawl is discussed in Feynman, R. P.; Leighton, R. B.; Sands, M. *The Feynman Lectures on Physics: Definitive Edition*. Addison Wesley: Reading, Mass, **1963** (Pearson Addison-Wesley: San Francisco, Calif., **2006**); Vol. I, Chap. 46.
More recent analyses of Feynman's classic system are developed in, for example:
(b) Parrondo, J. M. R.; Español, P. Criticism of Feynman's analysis of the ratchet as an engine. *Am. J. Phys.* **1996**, *64*, 1125–1130.
(c) Magnasco, M. O.; Stolovitzky, G. Feynman's Ratchet and Pawl. *J. Stat. Phys.* **1998**, *93*, 615–632.
(d) Jarzynski, C.; Mazonka, O. Feynman's ratchet and pawl: An exactly solvable model. *Phys. Rev. E* **1999**, *59*, 6448–6459.
(e) Sokolov, I. M. Reversible fluctuation rectifier. *Phys Rev. E* **1999**, *60*, 4446–4449.
(f) Velasco, S.; Roco, J. M. M.; Medina, A.; Hernández, A. C. Feynman's ratchet optimization: maximum power and maximum efficiency regimes. *J. Phys. D: Appl. Phys.* **2001**, *34*, 1000–1006.
(g) Pesz, K.; Gabrys, B. J.; Bartkiewicz, S. J. Analytical solution for the Feynman ratchet. *Phys. Rev. E* **2002**, *66*, 061103.
(h) Munakata, T; Suzuki, D. Rectification efficiency of the Feynman ratchet. *J. Phys. Soc. Japan* **2005**, *74*, 550–553.
(i) Suzuki, D.; Munakata, T. Stationary probability flow and vortices for the Smoluchowski-Feynman ratchet. *Physica A* **2006**, *368*, 16–24.
(j) Tu, Z. C. Efficiency at maximum power of Feynman's ratchet as a heat engine. *J. Phys. A - Math. Theor.* **2008**, *41*, Article Number 312003.
(k) For a survey (as of 1989) anticipating our velocity-dependent DP, see Denur, J. Velocity-dependent fluctuations: Breaking the randomness of Brownian motion. *Phys. Rev. A* **1989**, *40*, 5390–5399. Erratum. *Phys. Rev. A* **1990**, *41*, 3390. See Sect. I and Footnotes 1–5.
4. Reference [3a], the last paragraph of Sect. 46-2: This is the upshot of the discussions concerning Feynman's classic ratchet and pawl in Sect. 46-1 and 46-2. Said upshot is qualitatively justified on bases of (i) mechanics in Sect. 46-3 and (ii) special initial conditions in Sects. 46-4 and 46-5.
5. (a) Investigations of various formulations of the second law, various viewpoints concerning the absolute validity thereof, and challenges thereto in both the quantum and classical regimes, are given in Sheehan, D. P., Editor. *First International Conference on Quantum Limits to the Second Law, AIP Conference Proceedings (Vol. 643)*. AIP: Melville, New York, **2002**. Various authors cited in Footnotes [6] – [10] also have papers published in this *AIP Conference Proceedings (Vol. 643)*. Specific topics and/or models are studied in Ref. [3k] and in the references cited in Footnotes [6] – [11] and [27]; with Refs. [31] and [3k] being precursors, and Refs. [28a] and [28b] being shorter versions, of this present paper.
For more recent investigations concerning the topics mentioned in Ref. [5a], and related topics, see also the following book (Ref. [5b]) and three Special Issues of journals (Refs. [5c] – [5e]):
(b) Čápek, V.; Sheehan, D. P. *Challenges to the Second Law of Thermodynamics: Theory and Experiment*. Springer: Dordrecht, **2005** (Fundamental Theories of Physics Series, Vol. 146)
(c) Nikulov, A. V.; Sheehan, D. P. (Guest Editors). *Entropy* **March 2004**, *6 (Issue 1)*. Special Issue on Quantum Limits to the Second Law of Thermodynamics.
(d) Nieuwenhuizen, Th. M.; Keefe, D. P.; Špička, V. (Guest Editors). Proceedings of the International Conference Frontiers of Quantum and Mesoscopic Thermodynamics FQMT'04 (Prague, Czech Republic, **26–29 July 2004**). *Physica E* **October 2005**, *29 (Issues 1–2)*.

(e) Sheehan, D. P. (Guest Editor). *Found. Phys.* **December 2007**, *37 (Number 12)*. Special Issue: The Second Law of Thermodynamics: Foundations and Status.
6. Viewpoints concerning the second law in systems manifesting quantum-mechanical entanglement and/or coherence range from (I) that it can be violated: for example:
(a) Čápek, V.; Bok, J. Violation of the second law of thermodynamics in the quantum microworld. *Physica A* **2001**, *290*, 379–401.
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(c) Čápek, V. Zeroth and second laws of thermodynamics simultaneously questioned in the quantum microworld. *Eur. Phys. J. B* **2002**, *25*, 101–113.
(d) Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Breakdown of the Landauer bound for information erasure in the quantum regime. *Phys. Rev. E* **2001**, *64*, 056117.
(e) Nieuwenhuizen, Th. M.; Allahverdyan, A. E. Statistical thermodynamics of quantum Brownian motion: Construction of perpetuum mobile of the second kind. *Phys. Rev. E* **2002**, *66*, 036102.
(f) Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Bath-generated work extraction and inversion-free gain in two-level systems. *J. Phys. A Math. Gen.* **2003**, *36*, 875–882.
(g) Čápek, V. Dimer as a challenge to the second law. *Eur. Phys. J. B* **2003**, *34*, 219–223. Note: Reference [6g] was Dr. V. Čápek's last paper before he passed away. Therein, Dr. V. Čápek responds to Ref. [6l], and Dr. Daniel P. Sheehan has written a tribute to Dr. V. Čápek. There is also a tribute to Dr. V. Čápek in Ref. [5d]: Špička, V.; Nieuwenhuizen, Th. M.; Keefe, D. P.; Špička, V. (Guest Editors). In memoriam: Vladislav Čápek (1943–2002). Ref. [5d], vii–viii.
(h) Nieuwenhuizen, Th. M. Thermodynamics and small quantum systems. *J. Mod. Optics* **2003**, *50*, 2433–2441.
(i) Reference [5b]; Chap. 3, Sect. 4.6.
to (II) that it cannot: for example:
(j) Scully, M. O. Extracting Work from a Single Thermal Bath via Quantum Negentropy. *Phys. Rev. Lett.* **2001**, *87*, 220601.
(k) Hoffmann, K. H. Quantum thermodynamics. *Ann. der Phys.* **2001**, *10*, 79–88.
(l) Novotný, T. Investigation of apparent violation of the second law of thermodynamics in quantum transport studies. *Europhys. Lett.* **2002**, *59*, 648–654.
(m) Linke, H. Coherent Power Booster. *Science* **2003**, *299*, 841–842.
(n) Scully, M. O.; Zubairy, M. S.; Agarwal, G. S.; Walther, H. Extracting Work from a Single Heat Bath via Vanishing Quantum Coherence. *Science* **2003**, *299*, 862–864.
(o) Scully, M. O.; Rostovtsev, Y.; Sariyanni, Z-E.; and Zubairy, M. S. Using quantum erasure to exorcize Maxwell's demon: I. Concepts and context. Ref. [5d], 29–39.
(p) Rostovtsev, Y.; Sariyanni, Z-E.; Zubairy, M. S.; and Scully, M. O. Using quantum erasure to exorcize Maxwell's demon: II. Analysis. Ref. [5d], 40–46.
(q) Sariyanni, Z-E.; Rostovtsev, Y.; Zubairy, M. S.; and Scully, M. O. Using quantum erasure to exorcize Maxwell's demon: III. Implementation. Ref. [5d], 40–46.
(r) A qualitative summary was given in Weiss, P. Breaking the Law. *Science News* **Oct. 7, 2000** *158 (No. 15)*, pp. 234, 235, and 239.
For recent studies concerning the second law in the quantum regime, see, for example:

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(t) Allahverdyan, A. E.; Balian, R; Nieuwenhuizen, Th. M. Quantum thermodynamics: thermodynamics at the nanoscale. *Journal of Modern Optics* **2004**, *51*, 2703–2711.
(u) Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Fluctuations of work from quantum subensembles: The case against quantum work-fluctuation theorems. *Phys. Rev. E* **2005**, *71*, 066102.
(v) Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Minimal work principle: Proof and counterexamples. *Phys. Rev. E* **2005**, *71*, 046107.
(w) Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Adiabatic processes need not correspond to optimal work. Ref. [5d], 74–81.
(x) Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Explanation of the Gibbs paradox within the framework of quantum thermodynamics. *Phys. Rev. E* **2006**, *73*, 066119.
General overviews of the second law in the quantum regime are given in, for example:
(y) Reference [5d], 1–28.
(z) Schröder, H.; Teifel, J.; Mahler, G. Work and work fluctuations in quantum systems. *Eur. Phys. J. Special Topics* **2007**, *151*, 181–188.
(aa) Susskind, L.; Lindesay, J. *An Introduction to Black Holes, Information, and the String Theory Revolution: The Holographic Universe*. World Scientific: Singapore, **2005**.
(bb) Scully, R. F. and Scully, M. O.: *The Demon and the Quantum*, Wiley-VCH: New York, **2007**; especially Sects. 6.6 – 6.9 and Endnotes for Sects. 6.6 – 6.9.
The various formulations of the second law are not all of equal generality:
(cc) Various formulations of the second law are discussed in, for example: Ref. [5b], Sect. 1.2; and Ref. [6t].
(dd) Violations of *a* second law can fall short of violations of *the* second law — as per, for example: Ref. [6t], p. 2707; and Ref. [6y], the third paragraph of Sect. 1.1. According to Dr. Marlan O. Scully, private communications at the *36th Winter Colloquium on the Physics of Quantum Electronics (PQE-2006)*, **January 2–6, 2006** (see Refs. [6j], [6n], [6o], [6p], and [6q]), by the time of said *Colloquium*, viewpoints seemed to be converging towards the conclusion that the work required to prepare systems with quantum-mechanical entanglement, correlations, and/or coherence that can then, say, surpass the (theoretical-maximum *classical*) Carnot efficiency, equals or exceeds the extra-Carnot-efficiency work that can thereby be obtained. (This was confirmed in later private communications from Dr. Marlan O. Scully.): Thus, the extra-Carnot-efficiency work *per se* violates *a* second law [specifically, as per Ref. [5b], pp. 4–5, the Carnot theorem formulation (and possibly also the Efficiency and/or Heat Engines formulations) of the second law], but the *overall* process — *including the preparation work* — does *not* violate *the* second law [i.e., the Zhang formulation of the second law, enunciated in Ref. [1a] (and restated in the first two paragraphs of Sect. 1, with further discussions in the last four paragraphs of Sect. 3, of this present paper)]. Nevertheless, some questions concerning the validity of *the* second law — not merely *a* second law — in the quantum regime (e.g., for nonlinear systems coupled to a bath) may still be open: See, for example: Kim, I; Mahler, G. Quantum Brownian motion and the second law of thermodynamics. *Eur. Phys. J. B* **2006**, *54*, 405–414. Erratum, *Eur. Phys. J. B* **2007**, *56*, 279. Also, Kim, I.; Mahler, G. “The second law of thermodynamics in the quantum Brownian oscillator at an arbitrary temperature”, *Eur. Phys. J. B* **2007**, *60*, 401–408.
(ee) According to Th. M. Nieuwenhuizen, private communications at the Symposium: The Second Law of Thermodynamics: Foundations and Status on June 19–20, 2006 (part of the 87th Annual Meeting of the

Pacific Division of the AAAS on June 18–22, 2006): All formulations of the second law of thermodynamics are equivalent in the classical regime [with one exception that is not applicable insofar as this present paper is concerned): see Ref. [6gg]], and also for infinite systems in the quantum regime. However, they are not equivalent for a finite quantum system, even if in thermal contact with an infinite heat bath. (For a finite quantum system in thermal contact with a finite heat bath, the deviations from classical behavior are even larger.)

(ff) Allahverdyan, A. E.; Johal, R. S.; Nieuwenhuizen, Th. M. Work extremum principle: Structure and function of quantum heat engines. *arXiv.0709.4125v1 [cond-mat-stat-mech]* **26 Sep 2007**.

(gg) Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Minimal Work Principle and its Limits for Classical Systems. *arXiv:cond-mat/0607579 v1* **22 Jul 2006**. Also, Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Minimal-work principle and its limits for classical systems. *Phys. Rev. E* **2007**, *75*, 051124.

7. (a) Spontaneous rectification of thermal voltage fluctuations in diodes with very small capacitance at very low temperatures is studied in, for example, McFee, R. Self-Rectification in Diodes and the Second Law of Thermodynamics. *Am. J. Phys.* **1971**, *39*, 814–820.

Spontaneous rectification based on the Little-Parks effect is investigated in, for example:

(b) Nikulov, A. V. Quantum force in a superconductor. *Phys. Rev. B* **2001**, *64*, 012505.

(c) Nikulov, A. V. One of Possible Applications of High-Tc Superconductors. *arXiv:physics/0106020 v1* **6 Jun 2001**.

(d) Nikulov, A. V. Perpetuum Mobile without Emotions. Ref. [5a], pp. 207–213.

(e) Dubonos, S. V.; Kuznetsov, V. I.; Nikulov, A. V. Quantum voltage oscillations observed on segments of an inhomogeneous superconducting loop. *arXiv:cond-mat/0305337 v1* **15 May 2003**.

(f) Dubonos, S. V.; Kuznetsov, V. I.; Zhilyaev, I. N.; Nikulov, A. V.; Firsov, A. A. Observation of the External-ac-Current-Induced dc Voltage Proportional to the Steady Current in Superconducting Loops. *JETP Lett.* **2003**, *77*, 371–375.

(g) Aristov, V. V.; Nikulov, A. V. Quantum Power Source. Putting in Order of a Brownian Motion without Maxwell's Demon. *arXiv:cond-mat/0310073 v1* **3 Oct 2003**. In *Proceedings of SPIE* **2003**, *Vol. 5128 "Quantum Informatics"*, 148–156.

(h) Berger, J. Noise Rectification by superconducting loop with two weak links. *Phys. Rev. B* **2004**, *70*, 024524.

(i) Nikulov, A. About Peretuum Mobile without Emotions. Ref. [5a], pp. 207–212.

(j) Reference [5b], Sects. 4.2.3 and 4.4.

(k) Nikulov, A. V. Why the persistent power can be observed in mesoscopic quantum system. *arXiv:cond-mat/0404717 v1* **29 Apr 2004**.

(l) Nikulov, A. V. Quantum limits to the second law and a breach of a symmetry. *Invited Lecture at the Conference: "Frontiers of Quantum and Mesoscopic Thermodynamics"*, **26–29 July 2004**. Will be published In *Phys. Rev. E*. (E-print: Quantum limits to the second law and breach of symmetry. *arXiv:cond-mat/0505508 v1* **20 May 2005**.) Two related papers: (i) Nikulov, A. V. Differences between application of some basic principles of quantum mechanics on atomic and mesoscopic levels. *arXiv:cond-mat/0506653 v1* **24 June 2005**. (ii) Nikulov, A. Fundamental Differences between applications of some basic principles of quantum mechanics on atomic and mesoscopic levels. In Burra G. Sidearth, Antonio Alfonso-Faus, Marius J. Fullana, Editors. *Frontiers of Fundamental Physics: Eighth International Symposium. AIP Conference Proceedings (Vol. 905)*. AIP: Melville, New York, **2007**; pp. 117–119. (E-print: Fundamental Differences Between Application of Basic principles of Quantum Mechanics on Atomic and Higher Levels. *arXiv:0707.0827*

v1 **2007**.) {Note: These two related papers investigate possible difficulties with the *purely*-dynamical law of conservation of (angular) momentum associated with (anti)*thermo*-dynamical second-law-violating currents at *non*zero resistance in Little-Parks-effect circuits. Of course, the Little-Parks (anti)*thermo*-dynamical second-law-violating effect could still obtain even given *strict* conservation of *purely*-dynamical *total* (angular) momentum: Generation of counter-rotation in the surroundings with which a Little-Parks current's electrons interact could *strictly* conserve *total* (angular) momentum at essentially infinitesimal cost — which might be paid for by the Little-Parks effect itself — in kinetic energy imparted to said surroundings, if said surroundings are extremely massive compared with the combined masses of these electrons. A possible coupling mechanism might entail leakage of these electrons' wave packets into classically forbidden regions and hence into said surroundings, perhaps similarly to the mechanism discussed in Ref. [15] cited in *arXiv:cond-mat/0506653 v1* **24 June 2005** [Hirsch, J. E. The Lorentz force and superconductivity. *Phys. Lett. A* **2003**, *315*, 474–479 (see especially, on p. 478, the second paragraph and Fig. 4)]. Even almost infinitesimally weak coupling might suffice, because the kinetic energy thus imparted to the surroundings also need be almost infinitessimal. This topic is further investigated in later papers, e.g.: Aristov, A. A.; Nikulov, A. V. Could the EPR correlation be in superconducting structures?. Possibility of experimental verification. *arXiv:cond-mat/0604566* **25 Apr 2006**; and Nikulov, A. V. About Essence of the Wave Function on Atomic Level in Superconductors. *arXiv:0803.1840v1 [physics.gcn-ph]* **12 Mar 2008**.}

(m) Spontaneous rectification in nanoscale RLC circuits is studied in, for example, Allahverdyan, A. E.; Nieuwenhuizen, Th. M. Testing the violation of the Claussius inequality in nanoscale electric circuits. *Phys. Rev. B* **2002**, *66*, 115309.

(n) Results are discussed and various viewpoints through 2005 are summarized in Berger, J. The Chernogolovka experiment. Ref. [5d], 100–103.

Examples of more recent work are:

(o) Nikulov, A. Fundamental Differences Between Application of Basic Principles of Quantum Mechanics on Atomic and Higher Levels. *arXiv:0707:0827* **2007**. (Talk presented at the *Eighth International Symposium Frontiers of Fundamental Physics*, Madrid, **October 17–19, 2006**.)

(p) Gurtovoĭ, V. L.; Dubonos, S. V.; Karpii, S. V.; Nikulov, A. V.; Tulin, V. A. Contradiction between the Results of Observations of Resistance and Critical Current Quantum Oscillations in Asymmetric Superconducting Rings. *J. Exp. Thoer. Phys.* **2007**, *105*, 262–267.

(q) Dubonos, S. V.; Gurtovoi, V. L.; Nikulov, A. V.; Tulin, V. A. Can dc voltage proportional to the persistent current be observed on segment of asymmetric mesoscopic ring? *arXiv:0707.0807v1 [cond-mat.supr-con]* **5 Jul 2007**.

(r) Nikulov, A. V. The quantum challenge at the mesoscopic level. *arXiv:0707.4351v1 [cond-mat.supr-con]* **30 Jul 2007**.

(s) Burlakov, A. A.; Gurtovoĭ, V. L.; Dubonos, S. V.; Nikulov, A. V.; Tulin, V. A. Little-Parks effect in a system of asymmetric superconducting rings. *JETP Letters* **2007**, *86*, 517–521.

(t) Gurtovoĭ, V. L.; Dubonos, S. V.; Nikulov, A. V.; Osipov, N. N.; Tulin, V. A. Dependence of the Magnitude and Direction of the Persistent Current on the Magnetic Flux in Superconducting Rings. *J. Exp. Thoer. Phys.* **2007**, *105*, 1157–1173.

(u) Nikulov, A. V. About Essence of the Wave Function on Atomic Level and in Superconductors. *arXiv:0803.1840v1 [physics.gen-ph]* **12 Mar 2008**.

(v) Burlakov, A. A.; Gurtovoi, V. L.; Dubonos, S. V.; Nikulov, A. V.; Tulin, V. A. Observation of the Little-Parks Oscillations in a System of Asymmetric Superconducting Rings. *arXiv:0805.1223v1 [cond-*

mat.supr-con] **8 May 2008**.
(w) Nikulov, A. V. Bohm's quantum potential and quantum force in superconductor. *arXiv:0812.4118v1 [quant-ph]* **22 Dec 2008**.
8. (a) Proposed violations of the second law in plasma systems can be studied classically. See, for example, Sheehan D. P.; Means, J. D. Minimum requirement for second law violation: A paradox revisited. *Phys. Plasmas* **1998**, *5*, 2469–2471; and references cited therein, especially those by Sheehan, D. P.
(b) An alternative (classical) viewpoint given in Cruden, B. A. On the proposed second law paradox in a nonzero floating potential. *Phys. Plasmas* **2001**, *8*, 5323–5326.
This alternative viewpoint is reviewed in
(c) Reference [5b], Sect. 8.4.
(d) However, quantum-mechanical analyses may be more rigorous. See, for example, V. Čápek; Sheehan, D. P. Quantum mechanical model of a plasma system: a challenge to the second law of thermodynamics. *Physica A* **2002**, *304*, 461–479; and references cited therein.
(e) Reference [5b], Chap. 8.
9. Many proposed violations of the second law in low-density gas systems can be analyzed classically. Low-density gas-systems in gravitational fields are investigated in, for example:
(a) Sheehan, D. P.; Glick, J.; Duncan, T.; Langton, J. A.; Gagliardi, M. J.; Tobe, R. Phase Space Portraits of an Unresolved Gravitational Maxwell Demon. *Found. Phys.* **2002**, *32*, 441–462.
(b) Sheehan, D. P.; Glick, J.; Duncan, T.; Langton, J. A.; Gagliardi, M. J.; Tobe, R. Phase space analysis of a gravitationally-induced, steady-state nonequilibrium. *Phys. Scripta* **2002**, *65*, 430–437.
(c) Sheehan, D. P. A Classical Gravitational Second Law Paradox. Ref. [5a], pp. 391–396.
(d) Reference [5b], Chap. 6.
A critique is given in
(e) Wheeler, J. C. Resolution of a Classical Gravitational Second-Law Paradox. *Found. Phys.* **2004**, *34*, 1029–1062.
This critique is reviewed in
(f) Reference [5b], Sect. 6.2.5.
Field-free low-density gas systems are investigated in, for example:
(g) Miller, S. L. Insights into the Second-Law of Thermodynamics from Anisotropic Gas-Surface Interactions. Ref. [5e], 1660–1684.
(h) Denur, J. Speed-Dependent Weighting of the Maxwellian Distribution in Rarefied Gases: A Second-Law Paradox? Ref. [5e], 1685–1706. [Note: Reference [9h] considers a *speed*-dependent second-law paradox, in contrast with the *velocity*-dependent second-law challenge considered in this present paper.)
(i) Crosignani, B. and Di Porto, P. Random Fluctuations of Diathermal and Adiabatic Pistons. Ref. [5e], 1707–1715.
10. Proposed violations of the second law in solid-state systems are understandable classically. See, for example:
(a) Sheehan, D. P.; Putnam, A. R.; Wright, J. H. A Solid-State Maxwell Demon. *Found. Phys.* **2002**, *32*, 1557–1595.
A concise summary of Ref. [10a] is given in
(b) Weiss, P. Motor design flouts physical law. *Science News* **Nov. 2, 2002**, *162 (No. 18)*, 286.
More recent research, concerning a p-n junction solid-state oscillating motor, is discussed in

(c) Sheehan, D. P.; Wright, J. H.; Putnam, A. R.; and Perttu, E. K. Intrinsically biased, resonant NEMS-MEMS oscillator and the second law of thermodynamics. Ref. [5d], 87–99.
(d) The system investigated in Ref. [10c] is analyzed with respect to fundamental second-law considerations in Sheehan, D. P.; Gross, D. H. E. Extensivity and the thermodynamic limit: Why size really does matter. *Physica A* **2006**, *270*, 461–482.
11. Reference [3k]. The analyses in this present paper, as well as in previous shorter versions [28] thereof, are more quantitatively correct. Required conceptual and notational improvements, as well as numerical corrections, for Ref. [3k] as per this present paper (so that Ref. [3k] can be referred to more consistently with respect to this present paper), and related points, are discussed in Appendix I of this present paper. The analyses in this present paper (except for the last four paragraphs of Sect. 3, Sect. 6, and a few very brief mentions elsewhere, therein), as well as in Refs. [3k], [28a], and [28b], are classical.
12. (a) Reference [3k]. See especially the two paragraphs immediately following that containing Eq. (7), Appendixes A, B, and C, and Footnote 7. These discussions in Ref. [3k] are supplemented by Appendixes A, B, C, F, G, and H of this present paper (with conceptual and notational improvements, as well as numerical corrections, therein, and as noted in Appendix I of this present paper). The references cited in Footnote 7 of Ref. [3k] provide further supplementation.
An extension of the concept of temperature fluctuations is developed in, e.g.:
(b) Beck, C. Dynamical Foundations of Nonextensive Statistical Mechanics. *Phys. Rev. Lett.* **2001**, *87*, 180601.
(c) Beck, C. Superstatistics: Recent developments and applications. *arXiv:cond-mat/0502306 v1* **12 Feb 2005**. Also in Zichichi, A. (Senior Editor). Beck, C.; Benedek, G.; Rapisarda, A.; Tsallis, C. (Editors). *Complexity, Metastability, and Nonextensivity*. World Scientific: Singapore, **2005**; pp. 33–42.
(d) Abe, S.; Beck, C.; Cohen, E. G. D. Superstatistics, thermodynamics, and fluctuations. *Phys. Rev. E* **2007**, *76*, 031102
13. See, for example:
(a) Peebles, P. J. E., *Principles of Physical Cosmology*. Princeton University Press: Princeton, N. J., **1993**; pp. 134–138, 151–158, 174, and 176–181.
(b) Misner, C. W.; Thorne, K. S.; Wheeler, J. A. *Gravitation*. W. H. Freeman: New York, **1973**; Sect. 22.6 [especially pp. 587–589 and most especially Exercise 22.17 (of Chap. 22) on pp. 588–589].
(c) Peebles, P. J. E.; Wilkinson, D. T. Comment on the Anisotropy of the Primeval Fireball. *Phys. Rev.* **1968**, *174*, 2168.
(d) Gill, T. P. *The Doppler Effect*. Academic: New York, **1965**; Chap. VIII.
14. See, for example:
(a) Kolmogorov, A. N. *Foundations of the Theory of Probability, Second English Edition*. Chelsea: New York, **1956**.
(b) Ghahramani, S. *Fundamentals of Probability with Stochastic Processes, Third Edition*. Pearson Prentice Hall: Upper Saddle River, N. J., **2005**.
(c) Kelly, D. G. *Introduction to Probability*. Macmillan: New York, **1994**.
15. See, for example:
(a) Cox, D. R.; Miller, H. D. *The Theory of Stochastic Processes*. Chapman and Hall: London, **1965** (**1990** Printing)]; Sects. 3.1, 3.2, and 3.6.
(b) Jones, P. W.; Smith, P. *Stochastic Processes, An Introduction*. Arnold: London, **2001**; Chap. 4 (especially Sect. 4.3).

(c) The recursion method used in (10) is similar to that given in Hoel, P. G.; Port, S. C.; Stone, C. J. *Introduction to Stochastic Processes*. Houghton Mifflin: Boston, **1972** (reissued by Waveland: Prospect Heights, Ill., **1987**); pp. 1–2, Sects. 1.1, 1.2, and 1.4.2, and pp. 47–49.
16. See, for example, Nikulov, A. V. Ref. [7c] (see especially Sect. 4.2.); and Refs. [7d] and [7l].
17. Reference [3k], Appendixes B and C. Conceptual, notational, and numerical improvements, as per Appendix I of this present paper, are incorporated into discussions in Appendix B of this present paper, and into a review in Appendix F thereof concerning EBR radiant power, EBR radiant force, and thence the DP's Carnot-engine power outputs — in correction of Appendixes B and C of Ref. [3k].
18. See, for example, Leff, H. S.; Rex, A. F. in Leff, H. S.; Rex, A. F., Editors. *Maxwell's Demon 2: Entropy, Classical and Quantum Information, Computing*. Institute of Physics Publishing: Bristol, U. K., **2003**; Sect. 1.8.4 and references cited therein. This book contains an overview of various Maxwellian demons, reprints of many important papers in the field, and much additional informative material.
19. Reference [3k], the paragraph immediately following that containing Eq. (23), and the third paragraph of Appendix D.
20. To address the error in cited Ref. [19]: *Diffraction* does *not* reduce the opacity even of a small (linear dimensions $\lesssim hc/kT$) disk, or even of a very small (linear dimensions $\ll hc/kT$) disk: Note, for example, in correction, that the temperature of Doppler-shifted EBR impinging on a body is *un*affected by *diffraction*, even if size $\lesssim hc/kT$ (or even $\ll hc/kT$), as per:
(a) Reference [13a], pp. 174 and 176–177.
(b) Hecht, E. *Optics, Fourth Edition*. Addison-Wesley: New York, **2002**; Sect. 4.4.2, pp. 444–446, Sects. 10.4–10.5. (See especially p. 105 in Sect. 4.4.2.) Also: Hecht, E. Why don't Huygens' wavelets go backwards?. *Physics Teacher* **1980**, *18*, 149.
(c) Dr. Patricia H. Reiff, private communications, **1999**.
21. See, for example:
(a) Bohren, C. F.; Nevitt, T. J. Absorption by a sphere: a simple approximation. *Applied Optics* **1983**, *22*, 774–775.
(b) van de Hulst, H. C. *Light Scattering by Small Particles*. John Wiley: New York, **1957** (corrected republication by Dover: New York, **1981**); Chaps. 2, 6, and 14.
(c) Bohren, C. F.; Huffman, D. R. *Absorption and Scattering of Light by Small Particles*. John Wiley: New York, **1983** (Professional Paperback Edition, **1998**); Chap. 1, Chap. 5 (especially Sects. 5.1–5.2), and pp. 287–289.
(d) Reference [13a]; pp. 151–158, 174, and 176–181 (especially p. 174, and pp. 176–181).
(e) Some general discussions are given in Bohren, C. F.; Clothiaux, E. E. *Fundamentals of Atmospheric Radiation*. Wiley-VCH: Weinheim, **2006**; Sects. 1.3, 1.4, 2.9, and 3.5.
22. See, for example:
(a) Bohren, C. F. How can a particle absorb more than the light incident on it? *Am. J. Phys.* **1983**, *51*, 323–327.
(b) H. Paul; R. Fisher. Comment on "How can a particle absorb more than the light incident on it?". *Am. J. Phys.* **1983** *51*, 327.
(c) Bohren, C. F.; Huffman, D. R. Ref. [21c]; Sect. 3.4 (especially p. 72), Sect. 4.7 (especially the last two paragraphs — which show that resonance can obtain if size $\lesssim hc/kT$), and Chap. 12 (in Sect. 12.1.8, the model studied in Ref. [22a] is treated again).
(d) Garbuny, M. *Optical Physics*. Academic: New York, **1965**; pp. 100–101 and 268–269.

(e) Dr. Roland E. Allen, private communications, **2003**.
(f) References [20a] – [22e] may further address the difficulty associated with DP size $\lesssim hc/kT$ discussed qualitatively in Ref. [3k], the two paragraphs immediately following that containing Eq. (23) — noting (as per Footnote [20]), in correction, that the temperature of Doppler-shifted EBR impinging on a body is *un*affected by *diffraction*, even if DP size $\lesssim hc/kT$ (or even $\ll hc/kT$).
23. See, for example, Ref. [21b]; Sect. 11.23 (especially pp. 182–183), and p. 269.
24. See, for example, Ref. [21b], Chap. 14.
25. Reference [3k], Appendix D.
26. See, for example:
(a) Schiff, L. I. *Quantum Mechanics, Third Edition*. McGraw-Hill: New York, **1968**; pp. 100–107.
(b) Cohen, B. L. A Simple Treatment of Potential Barrier Penetration. *Am. J. Phys.* **1965**, *33*, 97–98.
27. (a) A challenge to the second law based on the Lorentz force was proposed in Fu, X. Y. An Approach to Realize Maxwell's Hypothesis. *Energy Convrs. Mgmt.* **1982**, *22*, 1–3.
This challenge was critiqued in:
(b) Wang,Y. R. Fu's Experiment and the Generalized Gibbs Distribution. *Energy Convrs. Mgmt.* **1983**, *23*, 185–191. {Note: Reference [27b] (see especially the first two paragraphs on p. 188) seems to imply that (in the *classical* regime) an increase in entropy is *not* equivalent to the Clausius-Heat formulation of the second law as stated in Ref. [5b], p. 4: "No process is possible for which the *sole* effect is that heat flows from a reservoir at a given temperature to a reservoir at a higher temperature.": ("sole" is not italicized in original text). But: If the entropy increase associated with the second term of Y. R. Wang's unnumbered equation [that immediately following his Eq. (12)] on p. 188 equals or exceeds in magnitude the entropy decrease associated with heat flowing "uphill" as per the first term thereof, then is not the Clausius-Heat formulation of the second law obeyed *despite* such "uphill" heat flow — since it is then *not* the only (or even the largest) effect? Does Y. R. Wang neglect the word *sole*?}
The Fu challenge seemed to be resolved in favor of the second law in:
(c) Wang, Y. Z. The Origin of the Electric Current in Fu's Experiment — An Analysis of Fu's Work on Realizing Maxwell's "Demon". *Energy Convrs. Mgmt.* **1986**, *26*, 249–252.
(d) Reference [1a]. Reference [1a] considers only (*non*dissipative) velocity-dependent forces acting perpendicularly to the velocity *itself* (i.e., to the direction of motion), and hence which can do no work — of which the Lorentz force is one example.
There have been more recent attempts to revive it:
(e) Fu, X.; Fu, Z. Realization of Maxwell's Hypothesis. *arXiv:physics/0311104* **2003**.
(f) Fu, X.; Fu, Z. Another Way to Realize Maxwell's Demon. *arXiv:physics/0509111* **2005**.
(g) As per private communications at the *36th Winter Colloquium on the Physics of Quantum Electronics (PQE-2006)*, **January 2–6, 2006**: The Fu analyses (in Refs. [27a], [27e], and [27f]) through the time of this **2006** *Colloquium* neglected electrons that leave plate B towards the right in *both* of Fu's Lorentz-force systems, as per *both* Fig. 1 *and* Fig. 2 of Ref. [27f]. This seemed, *at that time*, to resolve the Fu challenge in favor of the second law more strongly than does Ref. [27c]. But Fu met this challenge with a new Lorentz-force system [private communications at the *37th Winter Colloquium on the Physics of Quantum Electronics (PQE-2007)*, **January 2–6, 2007**]. Nevertheless, as per other private communications at this **2007** *Colloquium*, there may still be unresolved issues concerning even Fu's new Lorentz-force system.
(h) Note: The Zhang formulation of the second law, enunciated in Ref. [1a] (and restated in the first two paragraphs of Sect. 1, with further discussions in the last four paragraphs of Sect. 3, of this present paper),

is valid irrespective of whether or not (*non*dissipative) velocity-dependent forces acting perpendicularly to the velocity itself (i.e., to the direction of motion), and hence which can do no work — such as the Lorentz force — can challenge *the* second law (or even merely *a* second law): recall the last four paragraphs of Sect. 3, and Refs. [6s] – [6ff], especially Ref. [6dd].
28. (a) Denur, J. Modified Feynman ratchet with velocity-dependent fluctuations. Ref. [5a], pp. 326–331. A revised version of Ref. [28a] is:
(b) Denur, J. Modified Feynman ratchet with velocity-dependent fluctuations. Ref. [5c], 76–86. Corrections are given in Appendix J of this present paper.
29. Sheehan, D. P; Kriss, V. G. Energy Emission by Quantum Systems in an Expanding FRW Metric. *arXiv:astro-ph/0411299 v1* **11 Nov 2004**. See also references cited therein, especially Ref. 20 cited therein: Harrison, E. R. Mining Energy in an Expanding Universe. *Astroph. J.* **1995**, *446*, 63–66. [It should be noted, however, that it may be possible to consider the loss in the cosmic background radiation's kinetic energy associated with the cosmological redshift as being compensated for by the gain in its gravitational potential energy in the universe's gravitational field as the universe expands: This is easiest to visualize in a positively-curved (curvature index $= +1$) closed universe (shell of 3-sphere) wherein said radiation is trapped. Letting ν (λ) be the frequency (wavelength) corresponding to the peak of the cosmic-background-radiation blackbody Planck distribution, and considering expansion of said 3-sphere from radius of curvature R_1 to radius of curvature R_2, $\nu(R_2)/\nu(R_1) = \lambda(R_1)/\lambda(R_2) = R_1/R_2$. This implies conservation of *total — kinetic plus gravitational-potential* — energy given a cosmological gravitational potential difference of $\Delta\Phi = c^2 \ln \frac{R_2}{R_1}$. (Of course, similarly, during a possible contraction phase, it may be possible to consider the gain in the cosmic background radiation's kinetic energy associated with the cosmological blueshift as being compensated for by the loss in its gravitational potential energy in the universe's gravitational field as the universe contracts.) See, for example, Rindler, W. *Relativity: Special, General, and Cosmological, Second Edition*. Oxford: Oxford, U. K., **2006**; Sect. 1.6, Exercise 1.9 (of Chap. 1) on p. 28, Chap. 9, and Sects. 12.2 and 16.4.]
30. Scully, M. O.; Aharonov, Y.; Kapale, T. K.; Tannor, D. J.; Süssmann, G.; Walther, H. Sharpening accepted thermodynamic wisdom via quantum control: or cooling to an internal temperature of zero by external coherent control fields without spontaneous emission. *J. Mod. Optics* **2002**, *49*, 2297–2307.
31. (a) Denur, J. The Doppler demon. *Am. J. Phys.* **1981**, *49*, 352–355.
Constructive criticisms, given in
(b) Motz, H. The Doppler demon exorcised. *Am. J. Phys.* **1983**, *51*, 72–73;
and
(c) Chardin, G. No free lunch for the Doppler demon. *Am. J. Phys.* **1984**, *52*, 252–253
hopefully have been addressed in Ref. [3k], and more completely in this present paper and in previous shorter versions [28] thereof.
32. Reference [5b], Sect. 5.3.
33. See, for example, Reif, F. *Fundamentals of Statistical and Thermal Physics*. McGraw-Hill: New York, **1965**; Chap. 15. (See especially Sects. 15.5–15.8, and most especially the first complete paragraph on p. 573.)
34. Reference [13a]. To avoid confusion in comparison of notation between Ref. [13a] and this present paper: P. J. E. Peebles derives the EBR drag for an electron in the cosmic background radiation (which is, of course, an example of EBR), and expresses this EBR-drag result [Eq. (6.13) on p. 176] in terms of *Stefan's* constant a and the Thomson total cross-section for EBR scattering by an electron: see Peebles' remarks

concerning notation in the paragraph containing Eqs. (6.14) and (6.15) on pp. 137–138, in the last sentence on p. 158 [containing Eqs. (6.50 and (6.51)], in the first two sentences on p. 174 [containing Eq. (6.120], and on p. 176. By contrast, in this present paper, σ [$= \frac{c}{4}\times$ (Peebles' a)] is the *Stefan-Boltzmann* constant and $a =$ (cross-sectional area of disk) $= [\pi\times$ (radius of disk)^2 $-$ (cross-sectional area of guide)].

35. See, for example, Ref. [14b], Sect. 8.2, pp. 373–374, and Sects. 10.2–10.3.

36. The diminution of irreversibility of thermodynamic processes with increasing slowness of execution (corresponding to, e.g., approach to the Carnot efficiency for standard cyclic heat engines) obtains *typically but not universally*: Exceptions exist in the quantum regime (see Refs. [6s] – [6ff]), and one exception exists even in the classical regime (see Ref. [6gg]). Also, exceptions — albeit *un*related to extraction or expenditure of work and to this present paper — exist for thermodynamically-reversible (classical) cosmological models, which do *not* evolve infinitely slowly: see, for example, Tolman, R. C., *Relativity, Thermodynamics, and Cosmology*. Oxford University Press: Oxford, **1934** (unabridged and unaltered republication by Dover: New York, **1987**); Sects. 130 and 169–173.

37. The classical (quantum) case obtains if energy-level spacing $\ll kT$ ($\gg kT$). Considering (for simplicity) the *classical isothermal* case in this Footnote [37]: If x is the coordinate corresponding to *one* degree of freedom of interest, then, given a Boltzmann distribution and hence TEQ, we have, for the average thermal energy, $\langle E\rangle = \int_{x_{\min}}^{x_{\max}} xe^{-E(x)/kT}dx \div \int_{x_{\min}}^{x_{\max}} e^{-E(x)/kT}dx = \int_0^\infty Ee^{-E/kT}(dx/dE)dE \div \int_0^\infty e^{-E/kT}(dx/dE)dE$. If, for example, E is (or can at least be approximated as) $E \propto |x|^n$ ($n > 0$; of course, the absolute-value bars are superfluous for $x \geq 0$, and for all x if n is even and ≥ 2), then $\langle E\rangle = kT/n$. Typically, $n \approx 1$: e.g., $n = 1$ if E is potential energy of a particle in a uniform gravitational field; $n = 2$ if E is potential energy of simple harmonic oscillator or (nonrelativistic) kinetic energy, etc. $n \gg 1$ or $n \ll 1$ are rare, at least in typical naturally-occurring — especially atomic — systems. {As an aside, note that *even if* $0 < n \ll 1 \implies \langle E\rangle = kT/n \gg kT$ *does* obtain, this does *not* challenge the second law: $0 < n \ll 1 \implies \langle E\rangle = kT/n \gg kT$ implies a *very weak* tendency to restore the particle to $x = 0$, so that $(\mathrm{Var}\, x)^{1/2} \equiv \left[(x - \langle x\rangle)^2\right]^{1/2} = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2}$ is *very* large — and hence the expectation value of the work required to locate the particle is at least as large as $\langle E\rangle \gg kT$ that one expects to recover after locating it! [More correctly, said work must be expended via *memory erasure after locating the particle*: See, for example: Bennet, C. H. The Thermodynamics of Computation — a Review. *Int. J. Theor. Phys.* **1982**, *21*, 905–940 (especially Sect. 5). Also, Bennet, C. H. Demons, Engines, and the Second Law. *Sci. Am.* **November, 1987**, *257 (Number 5)*, 108–116 (cited references on pp. 150–151). The older viewpoint — that work must be expended to locate the particle — is reviewed in, for example, W. Ehrenberg, "Maxwell's Demon". *Sci. Am.* **November, 1967**, *217 (Number 5)*, 103–110 (cited references on p. 156). If, in locating the particle, $(\mathrm{Var}\, x)^{1/2}$ is reduced from an initial value of $(\mathrm{Var}\, x)_{\mathrm{i}}^{1/2}$ to a final value of $(\mathrm{Var}\, x)_{\mathrm{f}}^{1/2}$, then said work $= kT \ln \frac{(\mathrm{Var}\, x)_{\mathrm{i}}^{1/2}}{(\mathrm{Var}\, x)_{\mathrm{f}}^{1/2}} = T\times$ (negentropy cost) $=$ (free-energy cost).] An even more extreme (hypothetical) example of this is the potential energy of a test mass μ in a sufficiently weak gravitational field of an infinitely long cylinder of radius $x_{\min} > 0$ and linear mass density ρ_{linear} at TEQ in infinite three-dimensional Euclidean space, for which: (i) $E = 2G\rho_{\text{linear}}\mu \ln \frac{x}{x_{\min}}$ ($x =$ radial distance from central axis of cylinder) and (ii) sufficiently weak $\implies 0 < 2G\rho_{\text{linear}}\mu < kT$) — in which case $\langle E\rangle = \infty$, but $(\mathrm{Var}\, x)^{1/2} \equiv \left[(x - \langle x\rangle)^2\right]^{1/2} = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = e^{\infty}$!} In this Footnote [37], enclosure within (single) angular brackets simply denotes the averages indicated herein — *not* a time-dependent average relevant for our DP as per the third paragraph of Sect. 2 and the paragraph immediately following (24) of this present paper.

38. Reference [3k], Appendix D, third paragraph; with correction as per Appendix G [especially (G6)] of this present paper.
39. Reference [3k], Appendix D, first paragraph, with numerical corrections as per Item (c) of the last paragraph of Sect. 5 and the last three paragraphs of Appendix G of this present paper.
40. Dr. Daniel P. Sheehan, private communications, **2003–2005**; including a **2003** draft and the final **2005** version of Sect. 5.3 of Ref. [5b], that discusses Ref. [3k].